SO(10) A LA PATI-SALAM

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Abstract

We present rules for rewriting SO(10) tensor and spinor invariants in terms of invariants of its "Pati-Salam" maximal subgroup ($\mathrm{SU(4)}\times\mathrm{SU(2)_L}\times\mathrm{SU(2)_R}$) supplemented by the discrete symmetry called D parity. Explicit decompositions of quadratic and cubic invariants relevant to GUT model building are presented and the role of D parity in organizing the terms explained. Our rules provide a complete and explicit method for obtaining the Clebsch-Gordon Coefficients for $SO(10) \leftrightarrow G_{PS}$. We illustrate our methods by calculating mass matrices of SU(5) GUT type doublets and triplets in the minimal Susy SO(10) GUT. An extensive collection of $\mathrm{SO(6)} \leftrightarrow \mathrm{SU(4)}, \mathrm{SO(4)} \leftrightarrow \mathrm{SU(2)_L}\times\mathrm{SU(2)_R}$ translation identities is given.

I. INTRODUCTION

The virtues of SO(10) supersymmetric GUTs [1]- [7] are now widely appreciated. SO(10) has the cardinal virtue of exactly accommodating, within a single (16 dimensional) irrep, the 15 chiral fermions of a Standard Model family plus the right handed neutrino, which now has a strong claim to inclusion in any fundamental theory since neutrino masses are an inalienable part of particle phenomenology [8,9]. Thus the seesaw mechanism [10,11] finds a natural home in SO(10). Moreover SO(10) provides an appealing rationale for the parity breaking manifest in the Standard model by linking it to the breaking of Left-Right symmetry which embeds naturally in SO(10) via its Pati-Salam [12] maximal subgroup $G_{PS} = \mathrm{SU(4)}\times\mathrm{SU(2)_L}\times\mathrm{SU(2)_R}$ (More precisely $G_{PS}\times D$, where D is the so called D parity [13,14]).

There are, however, two contending points of view regarding the type of Higgs fields that should be used. Specifically, the question is whether [1]- [6], or not [7], large tensor representations like the 126 may be legitimately employed in view of their strong effect on the SO(10) beta function above the GUT scale and the difficulty of obtaining them from string theory. In supersymmetric models of the first type (which employ a "renormalizable see-saw mechanism" based on even B-L Higgs multiplets lying within the $\overline{\mathbf{126}}$ Higgs) the crucial R/M-parity of the MSSM becomes a part of the gauge symmetry and demonstrably survives symmetry breaking [6], [15]- [18]. In the alternative viewpoint [7] the use of $SO(10)$ spinorial $\mathbf{16}, \overline{\mathbf{16}}$ plet Higgs is advocated with nonrenormalizable couplings providing the

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effective $\overline{\mathbf{126}}$ dimensional operators needed for giving a large Majorana mass to the right handed neutrino. Other *ad hoc* symmetries are employed to play the role of R/M-parity which is strongly broken, obliterating the distinction between Higgs and sfermion scalars in the fundamental theory. This approach has the virtue of smaller threshold effects at the GUT scale and moreover the theory does not necessarily become asymptotically strong very close to the scale of perturbative Grand Unification. On the other hand it has recently been argued [19,20] that the explosion in the gauge coupling constant just above the GUT scale, due to the inclusion of Higgs multiplets adequate to achieve realistic tree level matter mass spectra, is in fact the flag of a new type of UV strong dynamical GUT symmetry breaking due to formation of SM singlet condensates , which can be analysed (since $M_{GUT} = M_U >> M_{Susy} = M_S$), using the methods (based on holomorphy of F-terms) developed by Seiberg and others [21] for supersymmetric gauge theories. In either type of theory knowledge of the Clebsch-Gordon coefficients for SO(10) or equivalently the ability to break up SO(10) invariants into those of its subgroups $G_{PS}, \supset G_{LR}, \supset G_{123}$ is essential.

In previous work [2,6,16,17,22] it was shown that in supersymmetric theories the restricted form of the superpotential can leave Renormalization Group (RG) significant multiplets with only intermediate or even light masses. Thus a proper RG analysis of Susy GUTs should make use of the actual mass spectrum of the model in question rather than the spectrum conjectured on the basis of the survival principle. To implement this program it is necessary to formulate the matching conditions for the couplings of the various mass multiplets at successive symmetry breaking and mass thresholds of the theory. Since the low energy theory is based upon a unitary gauge group whereas the ultimate determinant of coupling constant relations is the overlying SO(10) gauge symmetry it is necessary to write the SO(10) invariants in terms of properly normalized fields carrying the unitary maximal subgroup labels. The initial work on the minimal Susy GUT based on the 210-plet of SO(10) [2,3] was followed by an analysis of some of the SO(10) Clebsch-Gordon coeffcients in [23,24], which, however, could yield only incomplete results. The maximal subgroups of SO(10) are $SU(5) \times U(1)$ and the Pati-Salam Group $SU(4) \times SU(2)_L \times SU(2)_R$ which is isomorphic to the $SO(6) \times SO(4)$ subgroup of $SO(10)$. Very recently [26] the explicit forms of the SO(10) invariants of representations (with dimensions upto 210) were given in terms of $SU(5) \times U(1)$ labels using the so called oscillator basis [28] to effect the conversion. This rewriting, besides suffering from a certain lack of transparency (due precisely to the LR asymmetric nature of the embedding of $SU(5) \times U(1)$), is quite inappropriate for LR symmetric breaking chains. Thus it is necessary to obtain the invariants in terms of the PS subgroup separately. Moreover our results may be reassembled into $SU(5) \times U(1)$ invariants and can serve as an alternative derivation and cross check.

Furthermore, a discrete symmetry closely related to Parity [13], namely the so called D-parity, is important and useful in studying the possible symmetry breaking chains in SO(10) GUTs [6,14,29]. In the decomposition of SO(10) invariants into PS invariants D-parity proves valuable for organizing and cross checking relative signs in our expressions. We have developed explicit rules for the action of D-parity on all fields according to their (SO(10) tensor or spinor) origin and their PS labels.

Although the necessary basic tools have long existed (in somewhat implicit form) in the work of Wilczek and Zee [27] no explicit results are available. Moreover we disagree with [27] regarding the explicit form of the possible Charge conjugation matrices to be used for

$SO(2N)$ spinorial representations. Indeed it is only after making the necessary corrections that the translation $SO(10) \leftrightarrow G_{PS}$ becomes feasible and transparent. Therefore we have attempted to fill the long standing lacuna and provided rules for the translation from SO(10) labels to the PS unitary subgroup labels.

In Section II we introduce our notation and the embedding of SO(6)$\times$ SO(4) in SO(10) and define D-parity on tensor representations. We then show how to rewrite invariants formed from SO(6) tensor irreps in terms of SU(4) labels, and similarly for SO(4) invariants to $\mathrm{SU(2)_L \times SU(2)_R}$ labels. In Section III we implement these rules on some tensor invariants to illustrate the procedures for translating from SO(10) to G_{PS}. However, since an exhaustive listing of invariants is both exhausting to produce and counterproductive as regards actual utility for users of these techniques, we have instead provided an Appendix where we collect useful SO(6) and SO(4) contractions translated to unitary form. This collection permits easy computation of SO(10) invariants formed from any tensor representation of dimension ≤ 210. In Section IV, V we perform the same tasks once spinor representations are included. In Section VI we apply our results to compute doublet and triplet mass matrices in the Susy SO(10) GUT of [2,3,24,25]. We conclude with some remarks on future directions.

II. SO(10) $\rightarrow$ SO(6) $\times$ SO(4) $\sim G_{PS}$

The PS subgroup SU(4) $\times\mathrm{SU(2)_L \times SU(2)_R} \subset$ SO(10) is actually isomorphic to the obvious maximal subgroup SO(6) $\times$ SO(4) $\subset$ SO(10). The essential components of the analysis are thus explicit translation between SO(6) and SU(4) on the one hand and SO(4) and $\mathrm{SU}(2)_L \times \mathrm{SU(2)_R}$ on the other. Our notations and conventions follow those of [27] wherever possible. Wherever feasible we repeat definitions so that the presentation is self contained. A crucial difference with [27] concerning the explicit form of the charge conjugation matrices for spinor representations of orthogonal groups will however emerge in the section on spinors.

We have adopted the rule that any submultiplet of an SO(10) field is always denoted by the *same* symbol as its parent field, its identity being established by the indices it carries or by supplementary indices, if necessary. Our notation for indices is as follows : The indices of the vector representation of SO(10) (sometimes also SO(2N)) are denoted by $i,j = 1..10(2N)$. The *real* vector index of the upper left block embedding (i.e. the embedding specified by the breakup of the vector multiplet $10 = 6+4$) of SO(6) in SO(10) are denoted $a,b = 1,2..6$ and of the lower right block embedding of SO(4) in SO(10) by $\tilde{\alpha},\tilde{\beta} = 7,8,9,10$. These indices are complexified via a Unitary transformation and denoted by $\hat{a},\hat{b} = \hat{1},\hat{2},\hat{3},\hat{4},\hat{5},\hat{6} \equiv \overline{\mu},\overline{\mu}^* = \bar{1},\bar{1}^*,\bar{2},\bar{2}^*,\bar{3},\bar{3}^*$ where $\hat{1} \equiv \bar{1}, \hat{2} \equiv \bar{1}^*$ etc. Similarly we denote the complexified versions of $\tilde{\alpha},\tilde{\beta}$ by $\hat{\alpha},\hat{\beta} = \hat{7},\hat{8},\hat{9},\widehat{10}$. The indices of the doublet of $\mathrm{SU}(2)_L(\mathrm{SU(2)_R})$ are denoted $\alpha,\beta = 1,2(\dot{\alpha},\dot{\beta} = \dot{1},\dot{2})$. Finally the index of the fundamental 4-plet of SU(4) is denoted by a (lower) $\mu,\nu = 1,2,3,4$ and its upper-left block SU(3) subgroup indices are $\bar{\mu},\bar{\nu} = 1,2,3$. The corresponding indices on the 4^* are carried as superscripts.

A. SO(6) $\longleftrightarrow$ SU(4)

Vector/Antisymmetric: The 6 dimensional vector representation of SO(6) denoted by $V_a(a = 1, 2, .., 6)$ transforms as

$$V'_a = (exp\frac{i}{2}\omega^{cd}J_{cd})_{ab}V_b \tag{1}$$

where the Hermitian generators J_{cd} have the explicit form

$$(J_{cd})_{ef} = -i\delta_{c[e}\delta_{f]d} \tag{2}$$

and thus satisfy the SO(6) algebra (square brackets around indices denote antisymmetrization)

$$[J_{cd}, J_{ef}] = i\delta_{e[c}J_{d]f} - i\delta_{f[c}J_{d]e} \tag{3}$$

It is useful to introduce complex indices $\hat{a}, \hat{b} = \hat{1}...\hat{6}$ by the unitary change of basis

$$V_{\hat{a}} = U_{\hat{a}a}V_a \ , \quad U = U_2 \times I_3 \ , \quad U_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} \tag{4}$$

so that $V_aW_a = V_{\hat{a}}W_{\hat{a}^*}$. The decomposition of the fundamental 4-plet of SU(4) w.r.t. SU(3)$\times$U(1)$_{\mathrm{B-L}}$ is $4 = (3, 1/3) \oplus (1, -1)$. The index for the 4 of SU(4) is denoted by $\mu = 1, 2, 3, 4$ while $\bar{\mu} = 1, 2, 3$ label its SU(3) subgroup. In SU(4) labels, the 6 of SO(6) is the 2 index antisymmetric $V_{\mu\nu}$ and decomposes as $6 = V_{\bar{\mu}}(3, -2/3) \oplus V_{\bar{\mu}^*}(\bar{3}, 2/3)$ and we identify $V_{\bar{\mu}4} = V_{\bar{\mu}}$, $V_{\bar{\mu}\bar{\nu}} = \epsilon_{\bar{\mu}\bar{\nu}\bar{\lambda}}V_{\bar{\lambda}^*}$. In other words, if one defines $V_{\mu\nu} = \Theta^{\hat{a}}_{\mu\nu}V_{\hat{a}}$ with $\Theta^{\hat{a}}_{\bar{\mu}4} = \delta^{\hat{a}}_{\bar{\mu}}, \Theta^{\hat{a}}_{\bar{\mu}\bar{\nu}} = \epsilon_{\bar{\mu}\bar{\nu}\bar{\lambda}}\delta^{\hat{a}}_{\bar{\lambda}^*}$, then since $\Theta^{\hat{a}}_{\mu\nu}\Theta^{\hat{a}*}_{\lambda\sigma} \equiv \epsilon_{\mu\nu\lambda\sigma}$ it follows that the translation of SO(6) vector index contraction is ($\widetilde{V}^{\mu\nu} = (1/2)\epsilon^{\mu\nu\lambda\sigma}V_{\lambda\sigma}$)

$$V_aW_a = \frac{1}{4}\epsilon^{\mu\nu\lambda\sigma}V_{\mu\nu}W_{\lambda\sigma} \equiv \frac{1}{2}\widetilde{V}^{\mu\nu}W_{\mu\nu} \tag{5}$$

$$\text{while} \quad V_aW^*_a = \frac{1}{2}V_{\mu\nu}W^*_{\mu\nu} \tag{6}$$

Representations carrying vector indices a, b... are then translated by replacing by each vector index by an antisymmetrized pair of SU(4) indices $\mu_1\nu_1, \mu_2\nu_2$....... For example

$$A_{ab}B_{ab} = 2^{-4}\epsilon^{\mu_1\mu_2\mu_3\mu_4}\epsilon^{\nu_1\nu_2\nu_3\nu_4}A_{\mu_1\mu_2,\nu_1\nu_2}B_{\mu_3\mu_4,\nu_3\nu_4} \tag{7}$$

$$\text{while} \qquad A_{ab}B^*_{ab} = 2^{-2}A_{\mu_1\mu_2,\nu_1\nu_2}B^*_{\mu_1\mu_2,\nu_1\nu_2} \tag{8}$$

Antisymmetric/Adjoint: The 15 dimensional antisymmetric representation A_{ab} of SO(6) translates to the adjoint **15** $A_\nu{}^\mu$ of SU(4):

$$A_\nu{}^\mu = +\frac{1}{4}\epsilon^{\mu\lambda\sigma\rho}A_{\lambda\sigma,\rho\nu} = -A^\mu{}_\nu \quad ; \quad A_{\mu\nu,\rho\sigma} = +\epsilon_{\lambda\mu\nu[\rho}A_{\sigma]}{}^\lambda \tag{9}$$

The parameters ω_{ab} of SO(6) are identified with those of SU(4) (θ^A, $A = 1...15$)

$$\omega_{ab} \to \omega_{\mu}{}^{\nu} = i\theta^A (\lambda^A)_{\mu}{}^{\nu} \tag{10}$$

Where $\lambda^A, A = 1..15$ are the Gellmann matrices for $SU(4)$ and the group element in in the fundamental is $exp(\frac{i\theta^A\lambda^A}{2})$. We define

$$A_{\nu}{}^{\mu} = \frac{i}{\sqrt{2}}(\lambda^A)_{\nu}{}^{\mu} A^A \ , \qquad (\lambda^A)_{\nu}{}^{\mu} \equiv \lambda^A{}_{\nu\mu} \tag{11}$$

Note that tracelessness $A_{\mu}{}^{\mu} = 0$ is ensured by antisymmetry of $A_{\mu\nu,\lambda\sigma}$ and symmetry of $\epsilon^{\mu\nu\lambda\sigma}$ under interchange of index pairs $\mu\nu$ and $\lambda\sigma$. The normalization relation

$$\begin{aligned}(A_{\nu}{}^{\mu}, A_{\sigma}{}^{\lambda}) &= \delta^{\lambda}_{\mu}\delta^{\nu}_{\sigma} - \frac{1}{4}\delta^{\nu}_{\mu}\delta^{\lambda}_{\sigma} \\ &= \frac{1}{2}((\lambda^A)_{\nu}{}^{\mu})^*(\lambda^A)_{\sigma}{}^{\lambda}\end{aligned} \tag{12}$$

follows if A_{ab}, A^A are of unit norm :

$$(A_{ab}, A_{cd}) = \delta_{a[c}\delta_{d]b} \ \ ; \ \ (A^A, A^B) = \delta^{AB} \tag{13}$$

We denote the trace over SO(6) vector indices a,b ... by "Tr" and over the SU(4) fundamental index $\mu\nu$... by "tr". Then

$$\begin{aligned}TrAB &= A_{ab}B_{ba} = 2A_{\nu}{}^{\mu}B_{\mu}{}^{\nu} = 2trAB \\ TrABC &= -trA[B, C]\end{aligned} \tag{14}$$

A notable point is that the invariant 6 index totally antisymmetric tensor of SO(6) leads to a distinct $SU(4)$ invariant involving the anti- commutator.

$$\epsilon_{abcdef}A_{ab}B_{cd}C_{ef} = -8i(trA\{B, C\}) \tag{15}$$

Symmetric traceless (20)/4 index mixed: The 20 dimensional symmetric traceless representation S_{ab} of SO(6) which has normalization

$$(S_{ab}, S_{cd}) = \delta^a_{(c}\delta^b_{d)} - \frac{1}{3}\delta^{ab}\delta_{cd} \tag{16}$$

appropriate to a traceless field translates to $S_{\mu\nu,\lambda\sigma} = S_{\lambda\sigma,\mu\nu}$ with the additional constraint (corresponding to tracelessness on SO(6) vector indices)

$$\frac{1}{4}\epsilon^{\mu\nu\lambda\sigma}S_{\mu\nu,\lambda\sigma} \equiv S_{aa} = 0 \tag{17}$$

The normalization condition translates to

$$(S_{\mu\nu,\lambda\sigma}, S_{\theta\delta,\epsilon\rho}) = \delta^{\mu}_{[\theta}\delta^{\nu}_{\delta]}\delta^{\lambda}_{[\epsilon}\delta^{\sigma}_{\rho]} + \delta^{\mu}_{[\epsilon}\delta^{\nu}_{\rho]}\delta^{\lambda}_{[\theta}\delta^{\sigma}_{\delta]} - \frac{1}{3}\epsilon^{\mu\nu\lambda\sigma}\epsilon_{\theta\delta\epsilon\rho} \tag{18}$$

3 Index Antisymmetric (Anti) Self Dual/Symmetric 2 index: The invariant tensor ϵ_{abcdef} of SO(6) allows the separation of the 3 index totally antisymmetric 20-plet T_{abc} of

SO(6) into self dual and anti-self dual pieces $T^{\pm}_{abc} = \pm\widetilde{T}^{\pm}_{abc}$ where the SO(6) dual is defined as

$$\widetilde{T}_{abc} = \frac{i}{3!}\epsilon_{abcdef}T_{def} \tag{19}$$

$T^{+}_{abc}(T^{-}_{abc})$ translate into the 2 index symmetric $10(T_{\mu\nu})$ $(\overline{10}(\overline{T}^{\mu\nu}))$ of SU(4) via

$$T_{\mu\nu} = \frac{1}{12}T^{+}_{\mu\lambda,\nu\sigma,\gamma\delta}\epsilon^{\lambda\sigma\gamma\delta} \tag{20}$$

$$\overline{T}^{\mu\nu} = \frac{1}{24}T^{-}_{\kappa\lambda,\rho\sigma,\pi\theta}\epsilon^{\mu\kappa\lambda\pi}\epsilon^{\nu\rho\sigma\theta} \tag{21}$$

$$T^{(+)}_{\mu\nu,\rho\theta,\gamma\delta} = T_{[\mu[\rho}\epsilon_{\nu]\theta]\gamma\delta} \tag{22}$$

$$T^{(-)}_{\kappa\lambda,\theta\rho,\sigma\delta} = -\overline{T}^{\mu\nu}\epsilon_{\mu\kappa\lambda[\sigma}\epsilon_{\delta]\nu\theta\rho} \tag{23}$$

Note that to preserve unit norm one should define

$$T^{\pm}_{abc} = \frac{T_{abc} \pm \widetilde{T}_{abc}}{\sqrt{2}} \tag{24}$$

The normalization conditions that follow from unit norm for T_{abc} :

$$(T_{abc}, T_{a'b'c'}) = \delta^{a}_{[a'}\delta^{b}_{b'}\delta^{c}_{c']} \tag{25}$$

are

$$(T_{\mu\nu}, T_{\lambda\sigma}) = \delta^{\mu}_{(\lambda}\delta^{\nu}_{\sigma)} = (\overline{T}^{\lambda\sigma}, \overline{T}^{\mu\nu}) \tag{26}$$

So that $T_{\mu\mu}$(no sum) has norm squared 2 while $T_{\mu\nu}(\mu \neq \nu)$ has norm one.
One has the useful identity : $T^{+}_{abc}T^{-}_{abc} = 6\ T_{\mu\nu}\ \overline{T}^{\mu\nu}$

B. $\mathrm{SO(4)} \leftrightarrow \mathrm{SU(2)_L} \times \mathrm{SU(2)_R}$

Vector/Bidoublet

We use early greek indices $\tilde{\alpha}, \tilde{\beta} = 7, 8, 9, 10$ for the vector of SO(4) corresponding to $i, j = 7....10$ of the 10-plet of SO(10). The Hermitian generators of SO(4) have the usual SO(2N) vector representation form : $(J_{\tilde{\alpha}\tilde{\beta}})_{\tilde{\gamma}\tilde{\delta}} = -i\delta_{\tilde{\alpha}[\tilde{\gamma}}\delta_{\tilde{\delta}]\tilde{\beta}}$.
The group element is $R = exp\frac{i}{2}\omega^{\tilde{\alpha}\tilde{\beta}}J_{\tilde{\alpha}\tilde{\beta}}$. The generators of SO(4) separate neatly into self-dual and anti-self- dual sets of 3, $J^{\pm}_{\tilde{\alpha}\tilde{\beta}} = \frac{1}{2}(J_{\tilde{\alpha}\tilde{\beta}} \pm \tilde{J}_{\tilde{\alpha}\tilde{\beta}})$. Then if $\check{\alpha}, \check{\beta} = 1, 2, 3$ the generators and parameters of the $SU(2)_{\pm}$ subgroups of SO(4) are defined to be

$$J^{\pm}_{\check{\alpha}} = \frac{1}{2}\epsilon_{\check{\alpha}\check{\beta}\check{\gamma}}J^{\pm}_{(\check{\beta}+6)(\check{\gamma}+6)}\ ; \quad \omega^{\pm}_{\check{\alpha}} = \frac{1}{2}\epsilon_{\check{\alpha}\check{\beta}\check{\gamma}}\omega_{(\check{\beta}+6)(\check{\gamma}+6)} \pm \omega_{(\check{\alpha}+6)10} \tag{27}$$

The $SU(2)_{\pm}$ group elements are $exp(i\vec{\omega}_{\pm}\cdot\vec{J}^{\pm})$. The vector 4-plet of SO(4) is a bi-doublet $(2, 2)$ w.r.t. to $SU(2)_{-}\otimes SU(2)_{+}$. We denote the indices of the doublet of $SU(2)_L = SU(2)_{-}$

$(SU(2)_R = SU(2)_+)$ by undotted early greek indices $\alpha, \beta = 1, 2$ (dotted early greek indices $\dot\alpha, \dot\beta = \dot 1, \dot 2$). Then one has

$$V_{\hat 7} = V_{\bar 4} = \frac{(V_7 + iV_8)}{\sqrt 2} = V_{2\dot 2}\ ,\ \ V_{\hat 9} = V_{\bar 5} = \frac{(V_9 + iV_{10})}{\sqrt 2} = V_{1\dot 2} \tag{28}$$

$$V_{\hat 8} = V_{\bar 4^*} = \frac{(V_7 - iV_8)}{\sqrt 2} = -V_{1\dot 1}\ ,\ \ V_{\widehat{10}} \equiv V_{\hat 0} = V_{\bar 5^*} = \frac{(V_9 - iV_{10})}{\sqrt 2} = V_{2\dot 1} \tag{29}$$

$SU(2)_L(SU(2)_R)$ indices are raised and lowered with $\epsilon^{\alpha\beta}, \epsilon_{\alpha\beta}$ $(\epsilon^{\dot\alpha\dot\beta}, \epsilon_{\dot\alpha\dot\beta})$ with $\epsilon^{12} = +\epsilon_{21} = 1$ etc. The SO(4) vector index contraction translates as

$$V_{\tilde\alpha} W_{\tilde\alpha} = -V_{\alpha\dot\alpha} W_{\beta\dot\beta}\epsilon^{\alpha\beta}\epsilon^{\dot\alpha\dot\beta} = -V^{\alpha\dot\alpha}W_{\alpha\dot\alpha} \tag{30}$$

$$\text{While} \qquad V_{\tilde\alpha} W^*_{\tilde\alpha} = V_{\alpha\dot\alpha}W^*_{\alpha\dot\alpha} \tag{31}$$

Antisymmetric Selfdual/triplet : Separating the 2 index antisymmmetric tensor $A_{\tilde\alpha\tilde\beta}$ into self-dual and anti-self-dual parts of unit norm

$$A^{(\pm)}_{\tilde\alpha\tilde\beta} = \frac{1}{\sqrt 2}(A_{\tilde\alpha\tilde\beta} \pm \tilde A_{\tilde\alpha\tilde\beta}) \tag{32}$$

One finds $A^-(A^+)$ is $(3,1)((1,3))$ w.r.t. $SU(2)_L \times SU(2)_R$. In fact these triplets are just

$$\begin{aligned} A^{(\pm)}_{\check\alpha} &= \pm A^{(\pm)}_{\check\alpha+6,10} \\ &= \frac{1}{2}\epsilon_{\check\alpha\check\beta\check\gamma}A^{(\pm)}_{(\check\beta+6)(\check\gamma+6)} \end{aligned} \tag{33}$$

Defining $A_\alpha{}^\beta = iA^{(-)}_{\check\alpha}(\sigma^{\check\alpha})_\alpha{}^\beta = i\vec A_L \cdot (\vec\sigma)_\alpha{}^\beta$, $A_{\dot\alpha}{}^{\dot\beta} = iA^{(+)}_{\check\alpha}(\sigma^{\check\alpha})_{\dot\alpha}{}^{\dot\beta} = i\vec A^R \cdot (\vec\sigma)_{\dot\alpha}{}^{\dot\beta}$, where $\sigma^{\check\alpha}$ are the Pauli matrices, one has

$$A^{(+)}_{\hat\alpha\hat\beta} \to A^{(+)}_{\alpha\dot\alpha\beta\dot\beta} \equiv \epsilon_{\alpha\beta}A_{\dot\alpha\dot\beta} = \epsilon_{\alpha\beta}A_{\dot\beta\dot\alpha} \tag{34}$$

$$A^{(-)}_{\hat\alpha\hat\beta} \to A^{(-)}_{\alpha\dot\alpha\beta\dot\beta} \equiv \epsilon_{\dot\alpha\dot\beta}A_{\alpha\beta} = \epsilon_{\dot\alpha\dot\beta}A_{\beta\alpha} \tag{35}$$

Where the index pairs $\alpha\dot\alpha$ correspond to the complex indices $\hat\alpha$ as given in (29) above. Then one has for the contraction of two antisymmetric tensors

$$A_{\tilde\alpha\tilde\beta}B_{\tilde\alpha\tilde\beta} = \frac{1}{2}(A^{(+)}_{\tilde\alpha\tilde\beta}B^{(+)}_{\tilde\alpha\tilde\beta} + A^{(-)}_{\tilde\alpha\tilde\beta}B^{(-)}_{\tilde\alpha\tilde\beta}) \tag{36}$$

$$= 2(\vec A_L \cdot \vec B_L + \vec A_R \cdot \vec B_R) \tag{37}$$

Similarly one gets the useful identity

$$A^{(\pm)}_{\tilde\alpha\tilde\beta}B^{(\pm)}_{\tilde\beta\tilde\gamma}C^{(\pm)}_{\tilde\gamma\tilde\alpha} = 4\vec A^{(\pm)} \cdot (\vec B^{(\pm)} \times \vec C^{(\pm)}) \tag{38}$$

Symmetric Traceless(9)/Bitriplet(3,3) : The two index symmetric traceless tensor $S_{\tilde{\alpha}\tilde{\beta}}$ of SO(4) which has dimension 9 becomes the $(3,3)$ w.r.t $\mathrm{SU(2)_L} \times \mathrm{SU(2)_R}$ (symmetry follows from tracelessness):

$$S_{\hat{\alpha}\hat{\beta}} = S_{\alpha\dot{\alpha},\beta\dot{\beta}} \equiv S_{\alpha\beta,\dot{\alpha}\dot{\beta}} = S_{\beta\alpha,\dot{\alpha}\dot{\beta}} = S_{\alpha\beta,\dot{\beta}\dot{\alpha}} \tag{39}$$

so that e.g.

$$S_{\tilde{\alpha}\tilde{\beta}} S'_{\tilde{\alpha}\tilde{\beta}} = S^{\alpha\beta,\dot{\alpha}\dot{\beta}} S'_{\alpha\beta,\dot{\alpha}\dot{\beta}} \tag{40}$$

and are normalized as

$$(S_{\alpha\beta,\dot{\alpha}\dot{\beta}}, S_{\alpha'\beta',\dot{\alpha}'\dot{\beta}'}) = \delta^{\alpha}_{\alpha'}\delta^{\beta}_{\beta'}\delta^{\dot{\alpha}}_{\dot{\alpha}'}\delta^{\dot{\beta}}_{\dot{\beta}'} + \delta^{\alpha}_{\beta'}\delta^{\beta}_{\alpha'}\delta^{\dot{\alpha}}_{\dot{\beta}'}\delta^{\dot{\beta}}_{\dot{\alpha}'} - \frac{1}{2}\epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}\epsilon_{\alpha'\beta'}\epsilon_{\dot{\alpha}'\dot{\beta}'} \tag{41}$$

SO(10) Tensors & D-Parity

The above treatment covers the the SO(6) and SO(4) tensor representations encountered in dealing with SO(10) representations upto dimension 210. The procedure for the decomposition of SO(10) tensor invariants is now clear. Splitting the summation over each SO(10) index i,j= 1,..10 into summation over $SO(6), SO(4)$ indices (a, α), one replaces each SO(6) (SO(4)) index by $SU(4)(SU(2)_L \times SU(2)_R)$ index pair contractions according to the basic rules (5) and (31) and uses (9)(20)(21)(24) and (32)(34)(35) etc. to tranlate to PS labelled fields and invariants.
An important and useful feature of the decomposition is that it permits the transparent implementation of the Discrete symmetry called D-Parity [13,14] defined as

$$D = exp(-i\pi J_{23}) exp(i\pi J_{67}) \tag{42}$$

On vectors this corresponds to rotations through π in the (23) and (67) planes. Thus components(V_2, V_3, V_6, V_7) of V_i change sign and the rest do not. In PS language this becomes

$$V_{\mu\nu} \leftrightarrow (-)^{\mu+\nu+1}\widetilde{V}^{\mu\nu} \quad , \quad V_{2\dot{2}} \leftrightarrow V_{1\dot{1}} \tag{43}$$

While $V_{1\dot{2}}, V_{2\dot{1}}$ remain unchanged. If we denote $\bar{1} = 2$ and $\bar{2} = 1$ for dotted and undotted indices then these rules are just $V_{\alpha\dot{\beta}} \leftrightarrow V_{\dot{\bar{\beta}}\bar{\alpha}}$.

For the self-dual multiplets of SO(4) one finds that under D parity

$$V_1^{(\pm)} \leftrightarrow V_1^{(\mp)} \quad ; \quad V_{2,3}^{(\pm)} \leftrightarrow -V_{2,3}^{(\mp)} \tag{44}$$

I.e $V^{(-)}_{\alpha\beta} \leftrightarrow -V^{(+)}_{\dot{\bar{\alpha}}\dot{\bar{\beta}}}$. Then it follows that $\vec{A}_L \cdot \vec{B}_L \leftrightarrow \vec{A}_R \cdot \vec{B}_R$.
The adjoint ${A_\nu}^\mu$ derived from the antisymmetric 15 has D-parity property

$${A_\nu}^\mu \leftrightarrow (-)^{\mu+\nu+1} {A_\mu}^\nu \tag{45}$$

On the other hand an adjoint derived from a 4 index antisymmetric representation via

$$\Phi_{ab} = \frac{1}{4!}\epsilon_{abcdef}\Phi_{cdef} \tag{46}$$

as occurs, for example, for (15,1,1) $\subset$ 210 and (15,2,2) $\subset$ 126, $\overline{126}$, will contain an extra minus factor relative to (15,1,1) $\subset$ **45**. ${\phi_\nu}^\mu \leftrightarrow (-)^{\mu+\nu}{\phi_\mu}^\nu$ i.e. it is D-axial.
While the $SU(4)$ symmetric 10-plets from the SO(6) (anti)self-dual 3 index antisymmetric transform as

$$T_{\mu\nu} \leftrightarrow \overline{T}^{\mu\nu}(-)^{\mu+\nu+1} \tag{47}$$

III. SO(10) TENSOR QUADRATIC & CUBIC INVARIANTS

Using our rules we present examples of decompositions of SO(10) invariants to illustrate the application of our method. As noted above, however, the reader may find the generative rules collected in the Appendix more convenient and complete in practice.

<u>$45 \cdot 45$</u>

$$45(A_{ij}) = (15,1,1)A_{ab} + ((1,3,1)A^{(-)}_{\tilde{\alpha}\tilde{\beta}} \oplus (1,1,3)A^{(+)}_{\tilde{\alpha}\tilde{\beta}}) + (6,2,2)A_{a\tilde{\alpha}} \tag{48}$$

$$\begin{aligned} A_{ij}B_{ij} &= A_{ab}B_{ab} + 2A_{a\tilde{\alpha}}B_{a\tilde{\alpha}} + A_{\tilde{\alpha}\tilde{\beta}}B_{\tilde{\alpha}\tilde{\beta}} \\ &= -2A_{\nu}{}^{\mu}B_{\mu}{}^{\nu} - A^{\alpha\dot{\alpha}}{}_{\mu\nu}B^{\mu\nu}{}_{\alpha\dot{\alpha}} + 2(\vec{A}_L.\vec{B}_L + \vec{A}_R.\vec{B}_R) \end{aligned} \tag{49}$$

<u>$54 \cdot 54$</u>

$$54(S_{ij}) = (20,1,1,)\widehat{S}_{ab} + (1,3,3)\widehat{S}_{\tilde{\alpha}\tilde{\beta}} + (6,2,2)S_{a\tilde{\alpha}} + (1,1,1)S \tag{50}$$

$$S_{ij}R_{ij} = \widehat{S}_{ab}\widehat{R}_{ab} + \widehat{S}_{\tilde{\alpha}\tilde{\beta}}\widehat{R}_{\tilde{\alpha}\tilde{\beta}} + 2S_{a\tilde{\alpha}}R_{a\tilde{\alpha}} + 2S.R \tag{51}$$

$$= \frac{1}{4}\widehat{S}^{\mu\nu,\lambda\sigma}\widehat{R}_{\mu\nu,\lambda\sigma} + \widehat{S}^{\alpha\beta,\dot{\alpha}\dot{\beta}}\widehat{R}_{\alpha\beta,\dot{\alpha}\dot{\beta}} - S^{\mu\nu,\alpha\dot{\alpha}}R_{\mu\nu,\alpha\dot{\alpha}} + 2S.R \tag{52}$$

$$\text{where} \quad \widehat{S}_{ab} = S_{ab} - \sqrt{\frac{2}{15}}\delta_{ab}S \tag{53}$$

$$\widehat{S}_{\tilde{\alpha}\tilde{\beta}} = S_{\tilde{\alpha}\tilde{\beta}} + \sqrt{\frac{3}{10}}\delta_{\tilde{\alpha}\tilde{\beta}}S \tag{54}$$

$$S = \sqrt{\frac{5}{24}}S_{aa} \tag{55}$$

<u>$54 \cdot 54 \cdot 54$</u>

$$\begin{aligned} S_{ij}R_{jk}T_{ki} = &\frac{1}{2^3}\widehat{S}^{\mu\nu,\lambda\sigma}\widehat{R}_{\lambda\sigma}{}^{\theta\delta}\widehat{T}_{\theta\delta,\mu\nu} \\ &- \widehat{S}^{\alpha\beta,\dot{\alpha}\dot{\beta}}\widehat{R}_{\beta\gamma,\dot{\beta}\dot{\gamma}}\widehat{T}^{\gamma}{}_{\alpha},{}^{\dot{\gamma}}{}_{\dot{\alpha}} \\ &- \sqrt{\frac{2}{15}}S.R.T \\ &+ \sqrt{\frac{1}{120}}\{S\widehat{R}^{\mu\nu,\lambda\sigma}\widehat{T}_{\lambda\sigma,\mu\nu} + R\widehat{S}^{\mu\nu,\lambda\sigma}\widehat{T}_{\lambda\sigma,\mu\nu} + T\widehat{S}^{\mu\nu,\lambda\sigma}\widehat{R}_{\lambda\sigma,\mu\nu}\} \\ &- \frac{1}{4}\{\widehat{S}_{\mu\nu,\lambda\sigma}R^{\lambda\sigma}{}_{\alpha\dot{\alpha}}T^{\alpha\dot{\alpha},\mu\nu} + \widehat{R}_{\mu\nu,\lambda\sigma}T^{\lambda\sigma}{}_{\alpha\dot{\alpha}}S^{\alpha\dot{\alpha},\mu\nu} + \widehat{T}_{\mu\nu,\lambda\sigma}S^{\lambda\sigma}{}_{\alpha\dot{\alpha}}R^{\alpha\dot{\alpha},\mu\nu}\} \\ &- \sqrt{\frac{1}{120}}\{SR^{\mu\nu,\alpha\dot{\alpha}}T_{\mu\nu,\alpha\dot{\alpha}} + RT^{\mu\nu,\alpha\dot{\alpha}}S_{\mu\nu,\alpha\dot{\alpha}} + TS^{\mu\nu,\alpha\dot{\alpha}}R_{\mu\nu,\alpha\dot{\alpha}}\} \\ &+ \frac{1}{2}\{\widehat{S}^{\alpha\beta,\dot{\alpha}\dot{\beta}}R_{\beta\dot{\beta}}{}^{\mu\nu}T_{\mu\nu,\alpha\dot{\alpha}} + \widehat{R}^{\alpha\beta,\dot{\alpha}\dot{\beta}}T_{\beta\dot{\beta}}{}^{\mu\nu}S_{\mu\nu,\alpha\dot{\alpha}} + \widehat{T}^{\alpha\beta,\dot{\alpha}\dot{\beta}}S_{\beta\dot{\beta}}{}^{\mu\nu}R_{\mu\nu,\alpha\dot{\alpha}}\} \\ &- \sqrt{\frac{3}{10}}\{S\widehat{R}^{\alpha\beta,\dot{\alpha}\dot{\beta}}\widehat{T}_{\alpha\beta,\dot{\alpha}\dot{\beta}} + R\widehat{T}^{\alpha\beta,\dot{\alpha}\dot{\beta}}\widehat{S}_{\alpha\beta,\dot{\alpha}\dot{\beta}} + T\widehat{S}_{\alpha\beta,\dot{\alpha}\dot{\beta}}\widehat{R}_{\alpha\beta,\dot{\alpha}\dot{\beta}}\} \end{aligned} \tag{56}$$

$\underline{(45)^2 \cdot 54}$

$$
\begin{aligned}
A_{ij}A_{jk}S_{ki} &= 2A_\lambda{}^\mu A_\sigma{}^\nu \widehat{S}_{\mu\nu}^{\ \lambda\sigma} + \sqrt{\frac{8}{15}}A_\nu{}^\mu A_\mu{}^\nu S \\
&+ \sqrt{\frac{1}{30}}A^{\mu\nu,\alpha\dot{\alpha}}A_{\mu\nu,\alpha\dot{\alpha}}S + \frac{1}{4}A_{\mu\nu}{}^{\alpha\dot{\alpha}}A_{\lambda\sigma,\alpha\dot{\alpha}}\widehat{S}^{\mu\nu,\lambda\sigma} \\
&+ 2A^{\mu\nu,\alpha\dot{\alpha}}S_{\alpha\dot{\alpha},\lambda\mu}A_\nu{}^\lambda + \sqrt{\frac{1}{2}}A^{\mu\nu,\beta\dot{\beta}}(\epsilon_{\beta\alpha}A_{\dot{\beta}\dot{\alpha}} + \epsilon_{\dot{\beta}\dot{\alpha}}A_{\beta\alpha})S^{\alpha\dot{\alpha}}{}_{\mu\nu} \\
&- \sqrt{\frac{3}{40}}SA^{\mu\nu,\alpha\dot{\alpha}}A_{\mu\nu,\alpha\dot{\alpha}} - \frac{1}{2}\widehat{S}^{\alpha\beta,\dot{\alpha}\dot{\beta}}A^{\mu\nu}{}_{\alpha\dot{\alpha}}A_{\mu\nu,\beta\dot{\beta}} \\
&+ \sqrt{\frac{6}{5}}S(\vec{A}_L.\vec{A}_L + \vec{A}_R.\vec{A}_R) - 2A^{\dot{\alpha}\dot{\beta}}A^{\alpha\beta}\widehat{S}_{\beta\alpha,\dot{\beta}\dot{\alpha}}
\end{aligned} \tag{57}
$$

$\underline{\overline{126} \cdot 126}$

$$
\begin{aligned}
\frac{1}{5!}\Sigma^{(-)}_{i_1\ldots i_5}\Sigma^{(+)}_{i_1\ldots i_5} = \{&\widetilde{\Sigma}^{(-)\mu\nu}\Sigma^{(+)}_{\mu\nu} + 2\Sigma^{(-)\ \ \mu\alpha\dot{\alpha}}_{\nu}\Sigma^{(+)\nu}_{\mu\alpha\dot{\alpha}} \\
&+\vec{\Sigma}^{(-)}_{R\mu\nu}\cdot\vec{\Sigma}^{(+)\mu\nu}_{R} + \vec{\Sigma}^{(+)}_{L\mu\nu}\cdot\vec{\Sigma}^{(-)\mu\nu}_{L}\}
\end{aligned} \tag{58}
$$

Here $\Sigma^{(+)}(\mathbf{126})(\Sigma^{(-)}(\overline{\mathbf{126}}))$ is the self-dual (antiself-dual) 5 index totally antisymmetric representation and the dual is defined as (note the minus sign)

$$
\widetilde{\Sigma}_{i_1\ldots i_5} = -\frac{i}{5!}\epsilon_{i_1\ldots.i_{10}}\Sigma_{i_6\ldots i_{10}};\ \widetilde{\Sigma}^{(\pm)} = \pm\Sigma^{(\pm)} \tag{59}
$$

The SO(10) duality implies a correlation between the SO(6) and SO(4) dualities of the SU(4) decuplet $\mathrm{SU(2)_L} \times \mathrm{SU(2)_R}$ triplets :

$$
+ = (-,+) \oplus (+,-) \quad , \quad - = (+,+) \oplus (-,-) \tag{60}
$$

Where $(-,+)$ refers to $(\overline{10},1,3)$ and $(+,-)$ to $(10,3,1)$. So that, for example, Σ^+ has the decomposition

$$
\begin{aligned}
\Sigma^+(126) = &\Sigma^{(+)\mu}_{\nu}{}_{\alpha\dot{\alpha}}(15,2,2) + \vec{\Sigma}^{(+)}_{\mu\nu}{}_L(10,3,1) \\
&+ \vec{\Sigma}^{(+)\ \mu\nu}_{R}(\overline{10},1,3) + \Sigma^{(+)}_{\mu\nu}(6,1,1)
\end{aligned} \tag{61}
$$

While the $\Sigma^-(\overline{126})$ has the conjugate expansion.

$\underline{45 \cdot \overline{126} \cdot 126}$: An example of the non trivial action of D parity is given by the terms containing the (15,1,1) in the invariant $45 \cdot \overline{126} \cdot 126$.

$$
\begin{aligned}
\frac{1}{2(4!)}A_{a_1a_2}\Sigma^{(-)}_{a_1i_1..i_4}\Sigma^{(+)}_{a_2i_1..i_4} = &A_\nu{}^\mu(\Sigma^{(-)\lambda\ \alpha\dot{\alpha}}_{\mu}\Sigma^{(+)\nu}_{\lambda\ \ \alpha\dot{\alpha}} - \Sigma^{(-)\nu\ \alpha\dot{\alpha}}_{\lambda}\Sigma^{(+)\lambda}_{\mu\ \ \alpha\dot{\alpha}}) \\
&- \vec{\Sigma}^{(-)}_{\mu\nu R}\cdot A_\sigma{}^\nu\cdot\vec{\Sigma}^{(+)\sigma\mu}_{R} + \vec{\Sigma}^{(+)}_{\mu\nu L}\cdot A_\sigma{}^\nu\cdot\vec{\Sigma}^{(-)\sigma\mu}_{L} \\
&+ A_\nu{}^\mu\widetilde{\Sigma}^{(-)\nu\lambda}\Sigma^{(+)}_{\lambda\mu}
\end{aligned} \tag{62}
$$

Note the relative minus sign in the $(15,1,1)_A(15,2,2)_\pm(15,2,2)_\mp$ and $((10,3_\pm)(\overline{10},3_\pm)(15,1,1)_A)$ terms due to the property $a_\nu{}^\mu \xrightarrow{D} (-)^{\mu+\nu+1} a_\mu{}^\nu$. The terms containing $A_{\tilde\alpha\tilde\beta}$ are given by

$$
\begin{aligned}
\frac{1}{4!} A_{\tilde\alpha\tilde\beta} \Sigma^{(-)}_{\tilde\alpha i_1..i_4} \Sigma^{(+)}_{\tilde\beta i_1..i_4} &= \sqrt{2}\{\vec{A}_R \cdot (\vec{\Sigma}^{R(-)}_{\mu\nu} \times \vec{\Sigma}^{(+)\mu\nu}_R) + \vec{A}_L \cdot (\vec{\Sigma}^{\mu\nu(-)}_L \times \vec{\Sigma}^{(+)L}_{\mu\nu}) \\
&\quad - (A^{\dot\alpha\dot\beta} \Sigma_\nu^{(-)\mu\alpha}{}_{\dot\alpha} \Sigma^{(+)\nu}_\mu{}_{\alpha\dot\beta} + A^{\alpha\beta} \Sigma^{(-)\mu}_\nu{}_\alpha{}^{\dot\beta} \Sigma^{(+)\nu}_\mu{}_{\beta\dot\beta})\}
\end{aligned} \tag{63}
$$

The invariance under D parity of both terms follows from the rules (43,44) which imply

$$
\vec{A}_R \cdot (\vec{B}_R \times \vec{C}_R) \leftrightarrow \vec{A}_L \cdot (\vec{B}_L \times \vec{C}_L) \tag{64}
$$

IV. SPINOR REPRESENTATIONS

A. Generalities of SO(2N) Spinors

In the Wilczek and Zee [27] notation the γ matrices of the Clifford algebra of SO(2N), ${\gamma_i}^{(N)}$ are defined iteratively as direct products of Pauli matrices.

$$
\gamma_i^{(n+1)} = {\gamma_i}^{(n)} \otimes \tau_3, \quad n = 1.....N-1 \tag{65}
$$

$$
\gamma^{(n+1)}_{(2n+1)} = 1 \otimes \tau_1 \tag{66}
$$

$$
\gamma^{(n+1)}_{(2n+2)} = 1 \otimes \tau_2 \tag{67}
$$

starting with $\gamma_1^{(1)} = \tau_1$, $\gamma_2^{(1)} = \tau_2$. One also defines

$$
\gamma_F^{(N)} = (-i)^N \prod_{i=1}^{2N} \gamma_i^{(N)} \equiv \bigotimes_{i=1}^{N} (\tau_3)_i = \gamma_F^{(m)} \otimes \gamma_F^{(N-m)},\ m = 1, ...N-1 \tag{68}
$$

so that $\gamma_F^2 = 1$, $\gamma_F \gamma_i = -\gamma_i \gamma_F$. The generators of SO(2N) in the spinor representation are defined as $(i \neq j)$

$$
J_{ij} = -\frac{\sigma_{ij}}{2} = -\frac{i}{4}[\gamma_i, \gamma_j] \tag{69}
$$

A crucial point (where we disagree with equation (A19) of [27]) is the form of the charge conjugation matrix C. Equation A(19) of [27] appears to contradict equation A(11) of the same paper since $((-)^n \neq (-)^{\frac{n(n+1)}{2}}$ in general).
Recall that $\psi^T C \chi$ is a SO(2N) singlet when

$$
\sigma_{ij}^T C = -C \sigma_{ij} \tag{70}
$$

Two obvious possible (real) choices for C are

$$
C_1^{(n)} = \prod_{j=1}^{n} \gamma_{2j+1} \quad , \quad C_2^{(n)} = i^n \prod_{j=1}^{n} \gamma_{2j} \tag{71}
$$

$$\text{then} \qquad C_1^{(n)T} = (-)^{\frac{n(n-1)}{2}} C_1^{(n)} \quad , \quad C_2^{(n)T} = (-)^{\frac{n(n+1)}{2}} C_2^{(n)} \tag{72}$$

$$\gamma_i^T C_1 = (-)^{n-1} C_1 \gamma_i \quad , \quad \gamma_i^T C_2 = (-)^n C_2 \gamma_i \tag{73}$$

and both obey $C\gamma_F = (-)^n \gamma_F C$. Their explicit forms are easily obtained from

$$C_1^{(1)} = \tau_1 \ , \ C_2^{(1)} = i\tau_2 \tag{74}$$
$$C_1^{(n)} = \tau_1 \times C_2^{(n-1)} \tag{75}$$
$$C_2^{(n)} = i\tau_2 \times C_1^{(n-1)} \tag{76}$$

In particular $C_2^{(2m+1)} = i\tau_2 \times \bigotimes_{i=1}^m (\tau_1 \times i\tau_2)_i$ is clearly very different from eqn. A(19) of [27] which reads

$$C = i\tau_2 \times i\tau_2 \times i\tau_2 \times \cdots \tag{77}$$

and thus our charge conjugation matrices obey their eqn. A(11) (our eqn(72)) while (77) does not.
On chiral spinor irreps (projected using $(\frac{1\pm\gamma_F}{2})$) C_1 and C_2 are essentially equivalent. We shall define the SO(2N) charge conjugation matrix to be $C_2^{(N)}$. The Clifford algebra of SO(2N) acts on a 2^N dimensional space which is given the convenient basis of eigenvectors $|\epsilon = \pm 1 >$ of τ_3:

$$|\epsilon_1, ...\epsilon_n> = |\epsilon_1> \otimes ...... \otimes |\epsilon_n> \tag{78}$$

In this basis $\gamma_F = \prod_{i=1}^n \epsilon_i$. So the basis spinors of SO(2N) decompose into odd and even subspaces w.r.t. γ_F.

$$2^n = 2_+^{n-1} + 2_-^{n-1} \tag{79}$$

The SO(2N) dual of an N index object is

$$\widetilde{F}_{i_1.....i_N} = -\frac{i^N}{N!} \epsilon_{i_1......i_{2N}} F_{i_{N+1}...i_{2N}} \tag{80}$$

The identity

$$\gamma_{[i_1} ......\gamma_{i_M]} \gamma_F = \frac{(-i)^N (-)^{\frac{M(M-1)}{2}} M!}{(2N-M)!} \epsilon_{i_1......i_{2N}} \gamma_{i_{M+1}} ....\gamma_{i_{2N}} \tag{81}$$

is also frequently needed.

B. SO(6) Spinors

The $4(\psi_\mu)$ and $\bar{4}(\widehat{\psi}^\mu)$ of SU(4) may be consistently identified with the $4_-, 4_+$ chiral spinor multiplets of SO(6) by identifying components ψ_μ of the 4 with the coefficients of the states $|\epsilon_1\epsilon_2\epsilon_3>_-$ in $4_- = |\psi>_-$ as

$$|\psi>_{-}=\psi_1|-++>+\psi_2|+-+>+\psi_3|++->+\psi_4|--->\tag{82}$$

and also $\widehat{\psi}^\mu$ in the $4_+=|\widehat{\psi}>_+$ as

$$|\widehat{\psi}>_{+}=-\widehat{\psi}^1|+-->+\widehat{\psi}^2|-+->-\widehat{\psi}^3|--+>+\widehat{\psi}^4|+++>\tag{83}$$

The reason for the extra minus signs is that then the charge conjugation matrix $C_2^{(3)}$ correctly combines the $4,\bar{4}$ components in the 2^3-plet spinors of SO(6) to make SU(4) singlets and covariants . For example (we take ψ,χ to be non-chiral $8=4_++4_-$ spinors to preserve generality)

$$\psi^T\bar{C}_2^{(3)}\chi=\widehat{\psi}^\mu\chi_\mu+\psi_\mu\widehat{\chi}^\mu\tag{84}$$

$$\psi^\dagger\chi=\psi_\mu^*\chi_\mu+\widehat{\psi}^{\mu*}\widehat{\chi}^\mu\tag{85}$$

while

$$D^{\pm}_{abc}\equiv\frac{1}{3!}\psi^T_{\mp}C_2\gamma_{[a}\gamma_b\gamma_{c]}\chi_{\mp}=\pm\widetilde{D}^{\pm}_{abc}\tag{86}$$

$$\text{i.e}\qquad (4_-\times 4_-)_{\mathrm{self-dual}}\leftrightarrow 10\ \text{ of }\ \mathrm{SU}(4)\tag{87}$$

$$(4_+\times 4_+)_{anti.s.d}\leftrightarrow\overline{10}\ \ of\ \ SU(4)\tag{88}$$

Which is consistent with the identification $4_-\sim 4, 4_+\sim\bar{4}$ and the multiplication rules in SU(4). Transforming to the basis in which the components of the spinor $8=4_-+4_+$ are precisely the $4+\overline{4}$ i.e. $(\psi_\mu,\widehat{\psi}^\mu)$, one finds that in that basis

$$C_2^{(3)}=AntiDiag(I_4,I_4)\ ,\ C_1^{(3)}=AntiDiag(I_4,-I_4)\tag{89}$$

$$[\gamma_{\mu\nu}]=\begin{array}{c c} & \begin{array}{cc}\{\}^\sigma & \qquad\{\}_\sigma\end{array}\\ \begin{array}{c}\{\}_\lambda\\ \{\}^\lambda\end{array} & \begin{pmatrix}0 & \sqrt{2}\epsilon_{\mu\nu\lambda\sigma}\\ -\sqrt{2}\delta^\lambda_{[\mu}\delta^\sigma_{\nu]} & 0\end{pmatrix}\end{array}$$

In this basis one has in the 8 dimensional spinor rep. of SO(6)

$$exp(\frac{i\omega^{ab}J_{ab}}{2})=Diag(exp(\frac{i\theta^A\lambda^A}{2}),exp(\frac{-i\theta^A\lambda^{A*}}{2}))$$

when the parameters are related as in eqn(10). One finds the following useful identities hold

$$\begin{array}{ll}\psi^TC_2^{(3)}\chi & =\psi_\mu\widehat{\chi}^\mu+\widehat{\psi}^\mu\chi_\mu=\psi.\widehat{\chi}+\widehat{\psi}.\chi\\ \psi^TC_2^{(3)}\gamma_{\mu\nu}\chi & =\sqrt{2}(-\psi_{[\mu}\chi_{\nu]}+\widehat{\psi}^\lambda\widehat{\chi}^\sigma\epsilon_{\mu\nu\lambda\sigma})\\ \psi^TC_2^{(3)}\gamma_{\mu\nu}\gamma_{\lambda\sigma}\chi & =-2\{\widehat{\psi}^\theta\chi_{[\lambda}\epsilon_{\sigma]\mu\nu\theta}+\psi_{[\mu}\epsilon_{\nu]\lambda\sigma\theta}\widehat{\chi}^\theta\}\\ \psi^TC_2^{(3)}\gamma_{\mu\nu}\gamma_{\lambda\sigma}\gamma_{\theta\delta}\chi & =(\sqrt{2})^3\{\psi_{[\mu}\epsilon_{\nu]\lambda\sigma[\theta}\chi_{\delta]}+\widehat{\psi}^\omega\widehat{\chi}^\rho\epsilon_{\omega\mu\nu[\theta}\epsilon_{\delta]\rho\lambda\sigma}\}\end{array}\tag{90}$$

The results when $\psi^TC_2^{(3)}\to\psi^\dagger$ are obtained by the replacements $\psi_\mu\to\widehat{\psi}^{\mu*}$ and $\widehat{\psi}^\mu\to\psi^*_\mu$ on the R.H.S of all the identities in (90). The square root factors arise because the antisymmetric pair labels for the gamma matrices correspond to complex indices $\hat{a},\hat{b}$. Note that due to (81) one does not need the identities for more than 3 gamma matrices. See the appendix for useful translations of SO(6) spinor-tensor invariants calculable from these identities .

C. SO(4) Spinors

In the case of SO(4) the spinor representation is 4 dimensional and splits into $2_+ \oplus 2_-$. It is not hard to see that with the definitions adopted for the generators of $SU(2)_\pm$ the chiral spinors $2_\pm$ may be identified with the doublets $\psi_\alpha, \psi_{\dot\alpha}$ of $SU(2)_- = SU(2)_L$ and $SU(2)_+ = SU(2)_R$ as

$$|2>_- = |\psi>_- = \psi_1|+-> + \psi_2|-+>, \quad |2>_+ = |\psi>_+ = \psi_{\dot 1}|++> - \psi_{\dot 2}|--> \tag{91}$$

As in the SO(6) case one transforms to the unitary basis where $4 = 2_+ \oplus 2_-$ has components $(\psi_\alpha, \psi_{\dot\alpha})$. Then in that basis

$$C_2 = \begin{pmatrix} \epsilon^{\alpha\beta} & 0_2 \\ 0_2 & -\epsilon^{\dot\alpha\dot\beta} \end{pmatrix}\ ,\ C_1 = -\begin{pmatrix} \epsilon^{\alpha\beta} & 0_2 \\ 0_2 & \epsilon^{\dot\alpha\dot\beta} \end{pmatrix}\ ,\ [\gamma_{\rho\dot\rho}] = \sqrt{2}\begin{pmatrix} 0_2 & \epsilon_{\rho\alpha}\delta^{\dot\beta}_{\dot\rho} \\ \epsilon_{\dot\rho\dot\alpha}\delta^{\beta}_{\rho} & 0_2 \end{pmatrix} \tag{92}$$

The following expressions for spinor covariants then follow

$$\begin{aligned}
\psi^T C_2^{(2)}\chi &= \psi^{\dot\alpha}\chi_{\dot\alpha} - \psi^\alpha\chi_\alpha \\
\psi^T C_1^{(2)}\chi &= \psi^{\dot\alpha}\chi_{\dot\alpha} + \psi^\alpha\chi_\alpha \\
\psi^T C_2^{(2)}\gamma_{\alpha\dot\alpha}\chi &= \sqrt{2}(\psi_{\dot\alpha}\chi_\alpha - \psi_\alpha\chi_{\dot\alpha}) \\
\psi^T C_1^{(2)}\gamma_{\alpha\dot\alpha}\chi &= \sqrt{2}(\psi_{\dot\alpha}\chi_\alpha + \psi_\alpha\chi_{\dot\alpha}) \\
\psi^T C_2^{(2)}\gamma_{\alpha\dot\alpha}\gamma_{\beta\dot\beta}\chi &= 2\epsilon_{\dot\alpha\dot\beta}\psi_\alpha\chi_\beta - 2\epsilon_{\alpha\beta}\psi_{\dot\alpha}\chi_{\dot\beta} \\
\psi^T C_1^{(2)}\gamma_{\alpha\dot\alpha}\gamma_{\beta\dot\beta}\chi &= -2\epsilon_{\dot\alpha\dot\beta}\psi_\alpha\chi_\beta - 2\epsilon_{\alpha\beta}\psi_{\dot\alpha}\chi_{\dot\beta}
\end{aligned} \tag{93}$$

Furthermore

$$\begin{aligned}
\psi^\dagger\chi &= \psi^*_{\dot\alpha}\chi_{\dot\alpha} + \psi^*_\alpha\chi_\alpha \\
\psi^\dagger\gamma_{\alpha\dot\alpha}\chi &= -\sqrt{2}(\psi^{\alpha *}\chi_{\dot\alpha} + \psi^{\dot\alpha *}\chi_\alpha) \\
\psi^\dagger\gamma_{\alpha\dot\alpha}\gamma_{\beta\dot\beta}\chi &= 2\epsilon_{\dot\alpha\dot\beta}\psi^{\alpha *}\chi_\beta + 2\epsilon_{\alpha\beta}\psi^{\dot\alpha *}\chi_{\dot\beta}
\end{aligned} \tag{94}$$

Note that these can be obtained from the corresponding identities involving $C_1^{(2)}$ by the replacements $\psi^{\dot\alpha} \to \psi^*_{\dot\alpha}, \psi^\alpha \to \psi^*_\alpha$ or from the C_2 identities by $\psi^{\dot\alpha} \to \psi^*_{\dot\alpha}, \psi^\alpha \to -\psi^*_\alpha$.

D. SO(10) Spinors

The spinor representation of SO(10) is 2^5 dimensional and splits into chiral eigenstates with $\gamma_F = \pm 1$ as

$$2^5 = 2^4_+ + 2^4_- = 16_+ + 16_- \tag{95}$$

$$16 = 16_+ = (4_+, 2_+) + (4_-, 2_-) = (\overline{4}, 1, 2) + (4, 2, 1) \tag{96}$$

$$\overline{16} = 16_- = (4_+, 2_-) + (4_-, 2_+) = (\overline{4}, 2, 1) + (4, 1, 2) \tag{97}$$

Where the first equality follows from eqn(68) and second from the SO(6) to SU(4) and SO(4) to $SU(2)_L \times SU(2)_R$ translations: $4_- = 4, 2_+ = 2_R, 2_- = 2_L$. Thus we see that the SU(4) and $SU(2)_L \times SU(2)_R$ properties of the submultiplets within the $16, \overline{16}$ are strictly correlated. Use of the SO(6) and SO(4) spinor covariant identities allows fast construction of SO(10) spinor invariants. For example ,

$$\psi^T C_2^{(5)}\gamma_{\mu\nu}^{(5)}\chi = \psi^T (C_2^{(3)} \times C_1^{(2)})(\gamma_{\mu\nu}^{(3)} \times \tau_3 \times \tau_3)\chi = \psi^T (C_2^{(3)}\gamma_{\mu\nu}^{(3)} \times C_2^{(2)})\chi \tag{98}$$

Next one uses the identities (90,93) in parallel , keeping in mind that in the 16-plet the dotted ($\mathrm{SU(2)_R}$) spinors are always $\bar{4}$-plets of SU(4) and the undotted ones are 4-plets and vice versa for $\overline{16}$. When ψ, χ are both 16-plets one immediately reads off the result

$$\psi^T C_2^{(5)}\gamma_{\mu\nu}^{(5)}\chi = \sqrt{2}(\psi_{[\mu}^{\alpha}\chi_{\nu]\alpha} + \widehat{\psi}^{\lambda\dot{\alpha}}\widehat{\chi}_{\dot{\alpha}}^{\sigma}\epsilon_{\mu\nu\lambda\sigma}) \tag{99}$$

D parity on spinors : D parity acts on the spinors of SO(10) as

$$\begin{aligned} D_{spinor} &= e^{(-i\pi J_{23})}e^{(i\pi J_{67})} = -\gamma_2\gamma_3\gamma_6\gamma_7 \\ &= (\bigotimes_{i=1}^{3} i\tau_2) \times (i\tau_2 \times 1_2) = D^{(3)} \times D^{(2)} \end{aligned} \tag{100}$$

Thus the action of D factorizes. Under $D^{(3)}$ one interchanges spinors of opposite chirality as :

$$\widehat{\psi}^{\mu} \rightarrow (-)^{\mu+1}\psi_{\mu} \tag{101}$$

$$\psi_{\mu} \rightarrow (-)^{\mu}\widehat{\psi}^{\mu} \tag{102}$$

Similarly for $D^{(2)} = i\tau_2 \times 1$, one finds interchange

$$\psi_{\alpha} \rightarrow \psi_{\dot{\bar{\alpha}}}, \quad \psi_{\dot{\alpha}} \rightarrow -\psi_{\bar{\alpha}} \Rightarrow \psi^{\alpha} \rightarrow -\psi^{\dot{\bar{\alpha}}}, \psi^{\dot{\alpha}} \rightarrow +\psi^{\bar{\alpha}} \tag{103}$$

Where by $\bar{\alpha}$ we mean $\bar{1} = 2, \bar{2} = 1$. This implies the contraction of spinors $\psi_{\alpha}, \chi_{\dot{\alpha}}$ with a bidoublet $V_{\alpha\dot{\alpha}} = V_{\hat{a}}$ tranforms as

$$V^{\alpha\dot{\beta}}\psi_{\alpha}\chi_{\dot{\beta}} \rightarrow -V^{\bar{\beta}\dot{\bar{\alpha}}}\psi_{\dot{\bar{\alpha}}}\chi_{\bar{\beta}} \tag{104}$$

Similarly with $\mathrm{SU(2)_L}(\mathrm{SU(2)_R})$ vectors one gets

$$V_{(-)}^{\alpha\beta}\psi_{\alpha}\chi_{\beta} \leftrightarrow -V_{(+)}^{\dot{\bar{\alpha}}\dot{\bar{\beta}}}\psi_{\dot{\bar{\alpha}}}\chi_{\dot{\bar{\beta}}} \tag{105}$$

While

$$\psi^{\alpha}\chi_{\alpha} \leftrightarrow -\psi^{\dot{\alpha}}\chi_{\dot{\alpha}} \tag{106}$$

$$\widehat{\psi}^{\mu}\chi_{\mu} \leftrightarrow -\psi_{\mu}\widehat{\chi}^{\mu} \tag{107}$$

These rules are consistent with the action of D-parity on PS subreps SO(10) tensors derived earlier . Indeed one recovers them when one defines such tensors via bilinear covariants formed from SO(10) spinors.

SO(10) Spinor-Tensor Invariants

We next give the explicit decomposition of quadratic and cubic SO(10) invariants involving a pair ($16, 16$ or $16, \overline{16}$) of SO(10) spinors contracted with (the conjugate of) one of the tensors in their Kronecker product decomposition :

$$16 \otimes 16 = 10 \oplus 120 \oplus 126 \quad (108)$$
$$16 \otimes \overline{16} = 1 \oplus 45 \oplus 210 \quad (109)$$

Besides use of the spinor identities (90,93) the remainder of the task is merely to decompose the SO(10) index contractions into PS irrep. index contractions , take account of self-duality where relevant and maintain unit reference norm.

<u>**16 · 16 · 10**</u> : The 10-plet has decomposition: $H_i(10) = H_a(6,1,1) + H_{\tilde{\alpha}}(1,2,2)$ and one gets

$$\psi^T C_2^{(5)} \gamma_i^{(5)} \chi H_i = \sqrt{2}\{H_{\mu\nu}\widehat{\psi}^{\mu\dot{\alpha}}\widehat{\chi}^{\nu}_{\dot{\alpha}} + \widetilde{H}^{\mu\nu}\psi^{\alpha}_{\mu}\chi_{\nu\alpha} - H^{\alpha\dot{\alpha}}(\widehat{\psi}^{\mu}_{\dot{\alpha}}\chi_{\alpha\mu} + \psi_{\alpha\mu}\widehat{\chi}^{\mu}_{\dot{\alpha}})\} \quad (110)$$

Note how D parity is maintained by the interplay between the SO(6) and SO(4) sectors.

<u>**16 · 16 · 120**</u> : Since

$$\begin{aligned} O_{ijk}(120) &= O_{abc}(10+\overline{10},1,1) + O_{ab\tilde{\alpha}}(15,2,2) + O_{a\tilde{\alpha}\tilde{\beta}}((6,1,3)+(6,3,1)) + O_{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}}(1,2,2) \\ &= O^{(s)}_{\mu\nu}(10,1,1) + \overline{O}^{\mu\nu}_{(s)}(\overline{10},1,1) + O_{\nu\alpha\dot{\alpha}}{}^{\mu}(15,2,2) \\ &+ O^{(a)}_{\mu\nu\dot{\alpha}\dot{\beta}}(6,1,3) + O^{(a)}_{\mu\nu\ \alpha\beta}(6,3,1) + O_{\alpha\dot{\alpha}}(1,2,2) \end{aligned} \quad (111)$$

(where we have used the superscripts $^{(s),(a)}$ to discriminate the symmetric 10-plet from the antisymmetric 6-plet). Then one gets

$$\begin{aligned} \frac{1}{(3!)}\psi C_2^{(5)}\gamma_i\gamma_j\gamma_k\chi O_{ijk} &= -2(\bar{O}^{\mu\nu}_{(s)}\psi^{\alpha}_{\mu}\chi_{\nu\alpha} + O^{(s)}_{\mu\nu}\widehat{\psi}^{\mu\dot{\alpha}}\widehat{\chi}^{\nu}_{\dot{\alpha}}) \\ &- 2\sqrt{2}O_{\nu}{}^{\mu\alpha\dot{\alpha}}(\widehat{\psi}^{\nu}_{\dot{\alpha}}\chi_{\mu\alpha} - \psi_{\mu\alpha}\widehat{\chi}^{\nu}_{\dot{\alpha}}) \\ &- 2(O^{(a)\dot{\alpha}\dot{\beta}}_{\mu\nu}\widehat{\psi}^{\mu}_{\dot{\alpha}}\widehat{\chi}^{\nu}_{\dot{\beta}} + \widetilde{O}^{\mu\nu\alpha\beta}_{(a)}\psi_{\mu\alpha}\chi_{\nu\beta}) \\ &+ \sqrt{2}O^{\alpha\dot{\alpha}}(+\hat{\psi}^{\mu}_{\dot{\alpha}}\chi_{\mu\alpha} - \psi_{\mu\alpha}\widehat{\chi}^{\mu}_{\dot{\alpha}}) \end{aligned} \quad (112)$$

Note $O^{\alpha\dot{\alpha}}$ is derived from $O_{\tilde{\alpha}} = -\frac{1}{3!}\epsilon_{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}\tilde{\delta}}O_{\tilde{\beta}\tilde{\gamma}\tilde{\delta}}$ and so has opposite D parity to a vector $V_{\tilde{\alpha}}$.

<u>**16 · 16 · $\overline{\mathbf{126}}$**</u>

$$\overline{126} = \overline{\Sigma}^{(a)}_{\mu\nu}(6,1,1) + \overline{\Sigma}_{\nu}{}^{\mu}{}_{\alpha\dot{\alpha}}(15,2,2) + \overline{\Sigma}_{\mu\nu,\dot{\alpha}\dot{\beta}}(10,1,3) + \overline{\overline{\Sigma}}^{\mu\nu}{}_{\alpha\beta}(\overline{10},3,1) \quad (113)$$

$$\begin{aligned} \frac{1}{5!}\psi^T C_2^{(5)}\gamma_{i_1}.....\gamma_{i_5}\chi\overline{\Sigma}_{i_1...i_5} &= 2\sqrt{2}(\widetilde{\overline{\Sigma}}^{\mu\nu}_{(a)}\psi^{\alpha}_{\mu}\chi_{\nu\alpha} - \overline{\Sigma}^{(a)}_{\mu\nu}\widehat{\psi}^{\mu\dot{\alpha}}\widehat{\chi}^{\nu}_{\dot{\alpha}}) \\ &+ 4\sqrt{2}\overline{\Sigma}_{\nu}{}^{\mu\alpha\dot{\alpha}}(\widehat{\psi}^{\nu}_{\dot{\alpha}}\chi_{\alpha\mu} + \psi_{\mu\alpha}\widehat{\chi}^{\nu}_{\dot{\alpha}}) \\ &+ 4(\overline{\Sigma}^{\dot{\alpha}\dot{\beta}}_{\mu\nu}\widehat{\psi}^{\mu}_{\dot{\alpha}}\widehat{\chi}^{\nu}_{\dot{\beta}} + \overline{\overline{\Sigma}}^{\mu\nu,\alpha\beta}\psi_{\mu\alpha}\chi_{\nu\beta}) \end{aligned} \quad (114)$$

Here $(\overline{\Sigma}^{(a)}_{\mu\nu}) \leftrightarrow (-)^{\mu+\nu}\widetilde{\overline{\Sigma}}^{\mu\nu}_{(a)}$, $\overline{\Sigma}_{\mu}{}^{\nu} \leftrightarrow (-)^{\mu+\nu}\overline{\Sigma}_{\nu}{}^{\mu}$ have reversed D parity due to the dualization involved in their definition. We say a representation is D-Axial if due to dualization it has an extra minus sign in its D transformation relative to that expected from its tensor structure.

$\mathbf{16\cdot\overline{16}}$

$$16(\psi) = (4,2,1)\psi_{\mu\alpha} + (\bar{4},1,2)\hat{\psi}^{\mu}_{\dot{\alpha}} \tag{115}$$
$$\overline{16}(\overline{\chi}) = (\bar{4},2,1)\hat{\chi}^{\mu}_{\alpha} + (4,1,2)\chi_{\mu\dot{\alpha}} \tag{116}$$

$$\psi^T C_2^{(5)}\overline{\chi} = \widehat{\psi}^{\mu\dot{\alpha}}\chi_{\mu\dot{\alpha}} + \psi_{\mu\alpha}\widehat{\chi}^{\mu\alpha} = -\chi^T C_2^{(5)}\psi \tag{117}$$

$\mathbf{16\cdot\overline{16}\cdot 45}$

$$45 = A_{\nu}{}^{\mu}(15,1,1) + A_{\mu\nu,\alpha\dot{\alpha}}(6,2,2) + A_{\alpha\beta}(1,3,1) + A_{\dot{\alpha}\dot{\beta}}(1,1,3) \tag{118}$$

$$\begin{aligned}\frac{1}{(2!)}\psi^T C_2^{(5)}\gamma_i\gamma_j\overline{\chi}A_{ij} &= 2A_{\kappa}{}^{\mu}(-\psi^{\alpha}_{\mu}\widehat{\chi}^{\kappa}_{\alpha} + \widehat{\psi}^{\kappa\dot{\alpha}}\chi_{\mu\dot{\alpha}})\\ &- \sqrt{2}(A^{\dot{\alpha}\dot{\beta}}\widehat{\psi}^{\mu\dot{\alpha}}\chi_{\dot{\beta}\mu} + A^{\alpha\beta}\psi_{\mu\alpha}\widehat{\chi}^{\mu}_{\beta})\\ &- (\widetilde{A}^{\mu\nu,\alpha\dot{\alpha}}\psi_{\mu\alpha}\chi_{\nu\dot{\alpha}} + A_{\mu\nu}^{\ \ \alpha\dot{\alpha}}\widehat{\psi}^{\mu}_{\dot{\alpha}}\widehat{\chi}^{\nu}_{\alpha})\end{aligned} \tag{119}$$

$\mathbf{16\cdot\overline{16}\cdot 210}$:

$$\begin{aligned}210 &= \Phi_{\nu}{}^{\delta}(15,1,1) + \Phi_{\mu\nu,\alpha\dot{\alpha}}(10,2,2) + \overline{\Phi}^{\mu\nu}{}_{\alpha\dot{\alpha}}(\overline{10},2,2)\\ &+ \Phi_{\mu,\alpha\beta}^{\ \ \nu}(15,3,1) + \Phi_{\theta,\dot{\alpha}\dot{\beta}}^{\ \ \nu}(15,1,3) + \Phi(1,1,1)\end{aligned} \tag{120}$$

$$\begin{aligned}\frac{1}{(4!)}\psi^T C_2^{(5)}\gamma_{i_1}\ldots.\gamma_{i_4}\overline{\chi}\Phi_{i_1\ldots i_4} &= -2i\Phi_{\delta}{}^{\sigma}(\widehat{\psi}^{\delta\dot{\alpha}}\chi_{\sigma\dot{\alpha}} + \psi_{\sigma\alpha}\widehat{\chi}^{\delta}_{\alpha})\\ &+ 2\sqrt{2}(\overline{\Phi}^{\mu\nu,\alpha\dot{\alpha}}\psi_{\mu\alpha}\chi_{\nu\dot{\alpha}} + \Phi_{\mu\nu}{}^{\alpha\dot{\alpha}}\widehat{\psi}^{\mu}_{\dot{\alpha}}\widehat{\chi}^{\nu}_{\alpha})\\ &+ 2\sqrt{2}\{\Phi_{\delta}{}^{\mu,\alpha\beta}\psi_{\mu\alpha}\widehat{\chi}^{\delta}_{\beta} - \Phi_{\delta}{}^{\lambda\dot{\alpha}\dot{\beta}}\widehat{\psi}^{\delta}_{\dot{\alpha}}\chi_{\lambda\dot{\beta}}\}\\ &+ 2\{\widetilde{\Phi}^{\mu\nu,\alpha\dot{\alpha}}\psi_{\mu\alpha}\chi_{\nu\dot{\alpha}} + \Phi_{\mu\nu}{}^{\alpha\dot{\alpha}}\widehat{\psi}^{\mu}_{\dot{\alpha}}\widehat{\chi}^{\nu}_{\alpha}\}\\ &+\Phi(\psi^{\alpha}_{\mu}\widehat{\chi}^{\mu}_{\alpha} - \widehat{\psi}^{\mu\dot{\alpha}}\chi_{\mu\dot{\alpha}})\end{aligned} \tag{121}$$

$\Phi_{\nu}{}^{\mu}, \Phi$ are both D-Axial, while

$$D(\Phi_{\mu\nu}^{\ \ \alpha\dot{\beta}}) = (-)^{\mu+\nu+1}\overline{\Phi}^{\mu\nu\bar{\beta}\dot{\bar{\alpha}}} \tag{122}$$

Note that to obtain the results when 16^* is used instead of $\overline{16}$ one need only replace

$$\widehat{\chi}^{\mu\alpha} \to \chi^*_{\mu\alpha}, \quad \chi_{\mu}{}^{\dot{\alpha}} \to (\widehat{\chi}^{\mu}_{\dot{\alpha}})^* \tag{123}$$

because $C_2^{(5)} = C_2^{(3)} \times C_1^{(2)}$ (see the remarks following eqns(90,94). When calculating quartic invariants formed by contractions of SO(10) tensor covariants made from 16, $\overline{16}$ multiplets (which often arise in model building with non renormalizable superpotentials [7]) one need only apply the identities (90,93) after decomposing the SO(10) vector indices while treating one of the covariants as an operator with appropriate PS indices.

V. ILLUSTRATIVE APPLICATION

In this section we give some examples of the use of our methods for typical tasks that arise when studying GUTs. The translation of the SO(10) covariant derivatives may be seen from e.g.

$$\begin{aligned}\psi^\dagger(\partial+\frac{i}{2}A^{kl}g_uJ_{kl})\psi=\psi^*_{\mu\alpha}\partial\psi_{\mu\alpha}+\widehat{\psi}^{\mu*}_{\dot\alpha}\partial\widehat{\psi}^{\mu}{}_{\dot\alpha}\\
+ig_u\sqrt{2}\{\psi^*_{\kappa\alpha}A^A(\frac{\lambda^A}{2})_{\kappa\mu}\psi_{\mu\alpha}+\widehat{\psi}^{\mu*}{}_{\dot\alpha}A^A((\frac{-\lambda_A}{2})_{\mu\kappa})^*\widehat{\psi}^{\kappa}{}_{\dot\alpha}\\
+\widehat{\psi}^{\mu*}_{\dot\beta}(\frac{\vec{A}_R\cdot\vec\sigma}{2})_{\dot\beta}{}^{\dot\gamma}\widehat{\psi}_{\mu\dot\gamma}+\psi_{\mu\beta}{}^*(\frac{\vec{A}_L\cdot\vec\sigma}{2})_{\beta}{}^{\gamma}\widehat{\psi}_{\mu\gamma}\}\\
+\frac{g_u}{2}(\widehat{\psi}^{\nu*}_{\dot\alpha}\widetilde{A}^{\mu\nu\alpha}{}_{\dot\alpha}\psi_{\mu\alpha}+\psi^*_{\nu\alpha}A_{\mu\nu\alpha}{}^{\dot\alpha}\widehat{\psi}^{\mu}{}_{\dot\alpha})\end{aligned}\tag{124}$$

We see that Pati-Salam coupling constants emerge as $g_4=g_2=g_u\sqrt{2}$. The GUT generators $T^A,\vec{T}_R,\vec{T}_L$ are each normalized to 2 on the 16-plet and have $\sqrt{2}g_u$ as their associated coupling. In the vector representation covariant derivative behaves as

$$\begin{aligned}V^*_i(\partial+\frac{i}{2}g_uA^{kl}J_{kl})_{ij}V_j=\frac{1}{2}V^*_{\mu\nu}\partial V_{\mu\nu}+\frac{i}{2}g_u\sqrt{2}V^*_{\mu\nu}A^A(\frac{\lambda^A}{2})_{[\mu}{}^{\sigma}V_{\nu]\ \sigma}\\
+ig_u\sqrt{2}\overline{V}_{\alpha\dot\alpha}(\vec{W}_L\cdot(\frac{\vec\sigma}{2})_{\alpha}{}^{\beta}V_{\beta\dot\alpha}+\vec{W}_R\cdot(\frac{\vec\sigma}{2})_{\dot\alpha}{}^{\dot\beta}V_{\alpha\dot\beta})\end{aligned}\tag{125}$$

This can easily be adapted to decompose the kinetic terms of any of the tensor representations.

We further illustrate the application and utility of our methods by calculating two important mass matrices in the minimal Supersymmetric SO(10) GUT ([2], [3] [23–25]). A part, but not all, of these matrices was available earlier using the results of [24] on CG coefficients involving singlet subreps of SO(10). However our methods also allow calculations of CG coefficients that are not of the restricted class studied in [23,24]. The chiral supermultiplets of the model consist of a 210- plet Φ_{ijkl} responsible for breaking SO(10) down to $\mathrm{G}_{3211}=\mathrm{SU(3)_C\times SU(2)_L\times U(1)_R\times U(1)_{B-L}}$. A $\overline{126}(\overline{\Sigma}),126(\Sigma)$ pair is required to be present together to break $U(1)_R\times U(1)_{B-L}\to U(1)_Y$ while preserving Susy and is capable of generating realistic neutrino masses and mixings via the type I or type II seesaw mechanisms [10,11]. Moreover the SU(2) doublets in the $\overline{126}+126$ can also participate in the electroweak symmetry breaking. Finally there is a 10-plet containing $\mathrm{SU}(2)_L$ doublets and SU(3) triplets and 3 families of matter contained in 16-plets. The complete superpotential of this model is given by :

$$\begin{aligned}W=\frac{m}{2(4!)}\Phi_{ijkl}\Phi_{ijkl}+\frac{\lambda}{4!}\Phi_{ijkl}\Phi_{klmn}\Phi_{mnij}+\frac{M}{2(5!)}\Sigma_{ijklm}\overline{\Sigma}_{ijklm}\\
+\frac{\eta}{4!}\Phi_{ijkl}\Sigma_{ijmno}\overline{\Sigma}_{klmno}+\frac{1}{4!}H_i\Phi_{jklm}(\gamma\Sigma_{ijklm}+\overline{\gamma}\overline{\Sigma}_{ijklm})\\
+\frac{M_H}{2}H_i^2+h_{AB}\psi^T_AC_2^{(5)}\Gamma_i\psi_BH_i+\frac{f_{AB}}{5!}\psi^T_AC_2^{(5)}\gamma_{i_1}...\gamma_{i_5}\psi_B\overline{\Sigma}_{i_1...i_5}\end{aligned}\tag{126}$$

The GUT scale vevs that break the gauge symmetry down to the SM symmetry are [2,3]:

- i)

$$\langle(15,1,1)\rangle_{210} : \langle\phi_{abcd}\rangle = \frac{a}{2}\epsilon_{abcdef}\epsilon_{ef} \tag{127}$$

where $[\epsilon_{ef}] = Diag(\epsilon_2, \epsilon_2, \epsilon_2), \quad \epsilon_2 = i\tau_2$. Defining

$$\phi_{ab} \equiv \frac{1}{4!}\epsilon_{abcdef}\phi_{cdef} \tag{128}$$

We have in SU(4) notation $[\phi_\nu{}^\lambda]$ for the (15,1,1) and

$$[\langle\phi_\nu{}^\lambda\rangle] = \frac{ia}{2}Diag(I_3, -3) \equiv \frac{ia\Lambda}{2} \tag{129}$$

- ii)

$$\langle(15,1,3)\rangle_{210} \ : \ \langle\phi_{ab\tilde{\alpha}\tilde{\beta}}\rangle = \omega\epsilon_{ab}\epsilon_{\tilde{\alpha}\tilde{\beta}} \tag{130}$$

which translates to

$$\langle(\vec{\phi}^{(R)\nu}_{\mu\ \dot{1}\dot{2}})\rangle = -\frac{\omega\Lambda}{\sqrt{2}} \equiv i\langle(\vec{\phi}^{(R)\nu}_{\mu})_0\rangle \tag{131}$$

- iii)

$$\langle(1,1,1)\rangle_{210} \ : \ \langle\phi_{\alpha\beta\gamma\delta}\rangle = p\epsilon_{\alpha\beta\gamma\delta} \tag{132}$$

- iv)

$$\langle(10,1,3)\rangle_{\overline{126}} \ : \ \langle\overline{\Sigma}_{\hat{1}\hat{3}\hat{5}\hat{8}\hat{0}}\rangle = \bar{\sigma} = -i\langle\overline{\Sigma}^{(R)}_{44(+)}\rangle = \frac{\overline{\Sigma}_{44\dot{1}\dot{1}}}{\sqrt{2}} \tag{133}$$

- v)

$$\langle(\overline{10},1,3)\rangle_{126} \ : \ \langle\Sigma_{\hat{2}\hat{4}\hat{6}\hat{7}\hat{9}}\rangle = \sigma = i\langle\Sigma^{(R)44}_{(-)}\rangle = \frac{\Sigma^{44}_{\dot{2}\dot{2}}}{\sqrt{2}} \tag{134}$$

Under the SM gauge group G_{231} the 10 plet decomposes as

$$10 = H_\alpha(2,1,1) + \overline{H}_\alpha(2,1,-1) + t^{(1)}_{\bar{\mu}}(1,3,\frac{-2}{3}) + \bar{t}^{\bar{\mu}}_{(1)}(1,\bar{3},\frac{2}{3}) \tag{135}$$

which are the doublets and triplets familiar from SU(5) unification. In the case of SO(10) there are many other types of G_{321} multiplets beyond the ones encountered in the SU(5) case but we focus here only on the multiplets that can mix with the components of the 10-plet i.e. those that transform as H,$\overline{\text{H}}$,t or $\bar{\text{t}}$. The doublet $(2,1,\pm1)$ sector in fact consists of 4 pairs of doublets which are

$$h^{(1)}_\alpha = H_{\alpha \mathrm{i}}\ ,\ h^{(2)}_\alpha = \overline{\Sigma}_{\alpha \mathrm{i}}\ ,\ h^{(3)}_\alpha = \Sigma_{\alpha \mathrm{i}}\ ,\ h^{(4)}_\alpha = \Phi^{44\mathrm{i}}_\alpha \tag{136}$$

where $\Sigma_{\alpha\dot\alpha}, \overline{\Sigma}_{\alpha\dot\alpha}$ refer to the B-L singlet inside the (15,2,2) submultiplets of the 126,$\overline{126}$ and $h^{(4)}$ comes from the $(\overline{10}, 2, 2) \subset 210$. Similarly one has

$$\bar h^{(1)}_\alpha = H_{\alpha\dot 2}\ ,\ \bar h^{(2)}_\alpha = \overline{\Sigma}_{\alpha\dot 2}\ ,\ \bar h^{(3)}_\alpha = \Sigma_{\alpha\dot 2}\ ,\ \bar h^{(4)}_\alpha = \Phi^{\ \dot 2}_{44\alpha} \tag{137}$$

On the other hand, there are 5 pairs of triplets $\mathrm{t}(1, 3, -\frac{2}{3}), \bar{\mathrm{t}}(1, \bar 3, \frac{2}{3})$ that mix :

$$t^{(1)}_{\bar\mu} = H_{\bar\mu 4}\ ,\ t^{(2)}_{\bar\mu} = \overline{\Sigma}^{(a)}_{\bar\mu 4}\ ,\ t^{(3)}_{\bar\mu} = \Sigma^{(a)}_{\bar\mu 4}\ ,\ t^{(4)}_{\bar\mu} = (\vec{\overline{\Sigma}}^{(R)}_{4\bar\mu})_0\ ,\ t^{(5)}_{\bar\mu} = (\vec\Phi^{(R)4}_{\bar\mu})_{(-)} \tag{138}$$

$$\bar t^{\bar\mu}_{(1)} = \widetilde{H}^{\bar\mu 4}\ ,\ \bar t^{\bar\mu}_{(2)} = \overline{\Sigma}^{\bar\mu 4}_{(a)}\ ,\ \bar t^{\bar\mu}_{(3)} = \Sigma^{\bar\mu 4}_{(a)}\ ,\ \bar t^{\bar\mu}_{(4)} = (\vec\Sigma^{4\bar\mu}_{(R)})_0\ ,\ \bar t^{\bar\mu}_{(5)} = (\vec\Phi^{\ \bar\mu}_{4(R+)}) \tag{139}$$

Here $t^{(2)(3)}, \bar t^{(2)(3)}$ come from the (6,1,1) content of the $\overline{126}$ and 126 while $t^{(4)}, \bar t^{(4)}$ come from $(10, 1, 3)_{\overline{126}}$ and $(\overline{10}, 1, 3)_{126}$. Finally $t_{(5)}$ and $\bar{\mathrm{t}}_{(5)}$ come from $(15, 1, 3)_{210}$.
The GUT scale vevs described above give rise to mass matrices dependent only on the 7 parameters m,M,$M_H, \lambda, \eta, \gamma, \overline\gamma$. A fine tuning is then required to keep one pair of doublets light while all the other Higgs are superheavy. The feasibility of this fine tuning and the determination of the mixtures that stay light requires explicit calculation of these mass matrices. Our method allows straightforward and unambiguous calculation of these mass matrices (as well as all other submultiplet Clebsches).
The $h, \overline{h}$ mass matrix can be read off from the bilinear terms in the superpotential which have the structure $m_{ij}\overline{h}^{(i)\alpha} h^{(j)}_\alpha$. For example the 14 element involves $H_{\alpha\dot 2} \subset H_{\tilde\alpha}$and $\Phi^{44\ \mathrm{i}}_\alpha \subset (\overline{10}, 2, 2) \sim \phi^{(-)}_{\mathrm{abc}\tilde\alpha}$ and can receive a contribution only from the term $\overline\gamma\langle\overline\sigma\rangle\Phi H$ in W .i.e. from

$$\begin{aligned} -\frac{4\overline\gamma}{4!} H_{\tilde\alpha}\Phi_{abc\tilde\beta}\langle\overline\Sigma_{abc\tilde\alpha\tilde\beta}\rangle &= -\frac{1}{12} H_{\tilde\alpha}\Phi^{(-)}_{abc\tilde\beta}\langle\overline\Sigma^{(+,+)}_{abc\tilde\alpha\tilde\beta}\rangle = -\frac{\overline\gamma}{2}\Phi^{\mu\nu\dot\beta}_\alpha\langle\overline\Sigma^{(+,+)}_{\mu\nu\dot\alpha\dot\beta}\rangle H^{\alpha\dot\alpha} \\ &= -\frac{\sqrt2}{2}\overline\gamma H^{\alpha\mathrm{i}}\Phi^{44\mathrm{i}}\overline\sigma = -\frac{\overline\gamma}{\sqrt2}\overline\sigma\overline{h}^\alpha_{(1)} h_{(4)\alpha} \end{aligned} \tag{140}$$

In this way, by a routine use of the translation identities given in the text and in the appendix, one obtains the required "Clebsch-Gordon" coefficients without any ambiguity.

$$\begin{pmatrix} -M_H & +\overline\gamma\sqrt3(\omega-a) & -\gamma\sqrt3(\omega+a) & -\frac{\bar\gamma\bar\sigma}{\sqrt2} \\ -\overline\gamma\sqrt3(\omega+a) & 0 & -(M+4\eta(a+\omega)) & 0 \\ \gamma\sqrt3(\omega-a) & -(M+4\eta(a-\omega)) & 0 & -\eta\overline\sigma\sqrt6 \\ -\frac{\sigma\gamma}{\sqrt2} & -\eta\sigma\sqrt6 & 0 & -\frac{m}{2}+3\lambda(\omega-a) \end{pmatrix}$$

The element 43 and 24 are zero since they involve SU(4) contributions $\Phi^{(+)}\Sigma\langle\overline\Sigma^{(++)}\rangle$ and $\Phi^{(-)}\overline\Sigma\langle\Sigma^{(-+)}\rangle$ between two 10-plets or two $\overline{10}$-plets which vanish.
In a similar way one can calculate the triplet mass matrix

$$\begin{pmatrix} M_H & \overline\gamma(a+p) & \gamma(p-a) & 2\sqrt2 i\omega\bar\gamma & i\bar\sigma\bar\gamma \\ \bar\gamma(p-a) & 0 & M & 0 & 0 \\ \gamma(p+a) & M & 0 & 4\sqrt2 i\omega\eta & 2i\eta\overline\sigma \\ -2\sqrt2 i\omega\gamma & -4\sqrt2 i\omega\eta & 0 & M+2\eta p+2\eta a & -2\sqrt2\eta\overline\sigma \\ i\sigma\gamma & 2i\eta\sigma & 0 & 2\sqrt2\eta\sigma & -m-2\lambda(a+p+4\omega) \end{pmatrix}$$

A detailed analysis of the implications of the mass matrices of the minimal SO(10) GUT will appear elsewhere [31].

VI. DISCUSSION

In this paper we have carried out the tedious calculations required to provide a tool kit for ready translation of any SO(10) invariant one is likely to encounter in the course of SO(10) GUT model building into a convenient form where the fields carry unitary group labels. This allows calculation of all "Clebsch-Gordon" coefficients relevant to SO(10) GUT models : including many which were as yet unavailable in the literature. In addition we have obtained a very explicit description of the action of D parity on all fields. This allows one to follow the operation of D-parity, which implements Left-Right symmetry i.e. parity, in such theories. This translation is necessary in order to carry out RG analysis based on calculated mass spectra and will also be useful to obtain more accurate estimates of threshold uncertainties. We presented explicit examples of the use of these tools to calculate mass matrices etc. A systematic study of related theories along the lines of the program outlined in [25,29] using the tools developed here will be presented elsewhere [31]. We hope that our techniques and results will be found useful by other practitioners of the unwieldy and - so far - somewhat obscure art of SO(10) GUT building, even if only due to the simple minded and (perhaps) objectionably explicit approach we have taken to the analysis of this niggling group theoretical problem. Our rules may also be applied in other contexts where one encounters these groups for example in 10 dimensional field theories where the Lorentz group is SO(1,9) and a translation to SU(4) labels instead of SO(6) labels for the compactified sector may prove more convenient, specially for spinorial indices.

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APPENDIX

In this section we have collected useful $SO(6) \leftrightarrow SU(4)$, $SO(4) \leftrightarrow SU(2) \times SU(2)$ identities for the convenience of the reader while translating invariants of his choice using our methods.

A. SO(6)

Two vectors :

$$V_a W_a = \frac{1}{2}\tilde{V}^{\mu\nu} W_{\mu\nu} \qquad\qquad \tilde{V}^{\mu\nu} \equiv \frac{1}{2}\epsilon^{\mu\nu\lambda\sigma} V_{\lambda\sigma} \tag{A.1}$$

The "raised" versions of eqn. (9),(22),(23) are

$$A^{\mu\nu,\lambda\sigma} = +A_\theta{}^{[\mu}\epsilon^{\nu]\lambda\sigma\theta} \tag{A.2}$$

$$T_{(+)}^{\mu\nu,\lambda\sigma,\theta\delta} = -\epsilon^{\mu\nu\gamma[\theta}\epsilon^{\delta]\lambda\sigma\omega}T_{\gamma\omega} \tag{A.3}$$

$$T_{(-)}^{\mu\nu,\lambda\sigma,\theta\delta} = T^{[\mu_{[}\lambda}\epsilon^{\nu]}{}_{\sigma]}{}^{\theta\delta} \tag{A.4}$$

Two index antisymmetric tensors :

$$A_{ab}B_{ba} = 2A_\nu{}^\mu B_\mu{}^\nu \tag{A.5}$$

Two index traceless symmetric tensors

$$\widehat{S}_{ab}\widehat{R}_{ba} = \frac{1}{4}\widehat{S}^{\mu\nu,\lambda\sigma}\widehat{R}_{\mu\nu,\lambda\sigma} \tag{A.6}$$

Three index antisymmetric tensors :

$$T_{abc}U_{abc} = \frac{1}{2}(T^{+}_{abc}U^{-}_{abc} + T^{-}_{abc}U^{+}_{abc}) = 3(T_{\mu\nu}\overline{U}^{\mu\nu} + \overline{T}^{\mu\nu}U_{\mu\nu}) \tag{A.7}$$

where T^{+}_{abc}, T^{-}_{abc} are self-dual and anti-self-dual parts of T_{abc}.
Mixed two index and three index antisymmetric tensors :

$$A_{ab}T^{+}_{acd}U^{-}_{bcd} = -4(A_{\nu}{}^{\mu}T_{\mu\lambda}\overline{U}^{\nu\lambda}) \tag{A.8}$$

$$\phi_{abcd}T^{+}_{abe}U^{-}_{cde} = 8i\phi_{\nu}{}^{\mu}T_{\mu\lambda}\overline{U}^{\nu\lambda} \tag{A.9}$$

$$\epsilon_{abcdef}A_{ab}T^{+}_{cdg}U^{-}_{efg} = 16i(A_{\nu}{}^{\mu}T_{\mu\lambda}\overline{U}^{\nu\lambda}) \tag{A.10}$$

Three two index antisymmetric tensors

$$A_{ab}B_{bc}C_{ca} = -trA[B,C] \tag{A.11}$$

$$\epsilon_{abcdef}A_{ab}B_{cd}C_{ef} = -8itrA\{B,C\} \tag{A.12}$$

Three two index symmetric traceless tensors :

$$\widehat{S}_{ab}\widehat{R}_{bc}\widehat{T}_{ca} = \frac{1}{8}\widehat{S}^{\mu\nu,\lambda\sigma}\widehat{R}^{\theta\delta}_{\lambda\sigma}\widehat{T}_{\theta\delta,\mu\nu} \tag{A.13}$$

Two vectors and two index tensors :

$$Antisymmetric \;\; V_aW_bA_{ab} = V_{\mu\nu}W^{\nu\lambda}A_{\lambda}{}^{\mu} \tag{A.14}$$

$$Symmetric\ traceless \;\; V_aW_b\widehat{S}_{ab} = \frac{1}{4}\tilde{V}^{\mu\nu}\widetilde{W}^{\lambda\sigma}\widehat{S}_{\mu\nu,\lambda\sigma} \tag{A.15}$$

Vector with two index and three index antisymmetric tensors :

$$V_aA_{bc}T_{abc} = V_aA_{bc}\frac{(T^{+}_{abc} + T^{-}_{abc})}{\sqrt{2}} = \sqrt{2}(-\tilde{V}^{\mu\nu}A_{\nu}{}^{\lambda}T_{\mu\lambda} + V_{\mu\nu}A_{\lambda}{}^{\nu}\overline{T}^{\mu\lambda}) \tag{A.16}$$

$$\epsilon_{abcdef}V_aA_{bc}T_{def} = (i3!\sqrt{2})(\tilde{V}^{\mu\nu}A_{\nu}{}^{\lambda}T_{\mu\lambda} + V_{\mu\nu}A_{\lambda}{}^{\nu}\overline{T}^{\mu\lambda}) \tag{A.17}$$

Two antisymmetric (A,B) and one symmetric traceless (S) two index tensor :

$$A_{ab}\widehat{S}_{bc}B_{ca} = \frac{1}{2}A^{\mu}_{[\nu}\widehat{S}^{\lambda\delta}_{\lambda]\mu}B^{\nu}_{\delta} \tag{A.18}$$

For the product of a tensor with two antisymmetric indices and two tensors with two symmetric indices :

$$A_{ab}\widehat{S}_{bc}\widehat{R}_{ca} = \frac{1}{2}A_{\kappa}{}^{\mu}\widehat{S}^{\nu\kappa,\lambda\sigma}\widehat{R}_{\lambda\sigma,\mu\nu} \tag{A.19}$$

B. SO(6) Invariants with Spinors

For SO(6) sector $C \equiv C_2^{(3)}$

$$\psi^T C \gamma_a \chi V_a = \sqrt{2}(\widehat{\psi}^\mu \widehat{\chi}^\nu V_{\mu\nu} - \psi_\mu \chi_\nu \widetilde{V}^{\mu\nu}) \tag{A.20}$$

$$\psi^T C \gamma_a \gamma_b \chi V_a W_b = 2(\widehat{\psi}^\mu \chi_\nu V_{\mu\lambda} \widetilde{W}^{\nu\lambda} + \psi_\mu \widehat{\chi}^\lambda \widetilde{V}^{\mu\nu} W_{\lambda\nu}) \tag{A.21}$$

$$\psi^T C \gamma_a \gamma_b \chi A_{ab} = 4 A_\nu{}^\mu(-\psi_\mu \widehat{\chi}^\nu + \widehat{\psi}^\nu \chi_\mu) \tag{A.22}$$

$$\psi^T C \gamma_a \gamma_b \gamma_c \chi T_{abc} = 12(\overline{T}^{\mu\nu} \psi_\mu \chi_\nu - T_{\mu\nu} \widehat{\psi}^\mu \widehat{\chi}^\nu) \tag{A.23}$$

$$\psi^T C \gamma_a \gamma_b \gamma_c \chi V_a W_b U_c = 2\sqrt{2}(\psi_\mu \chi_\delta \tilde{V}^{\mu\nu} W_{\nu\theta} U^{\theta\delta} - \widehat{\psi}^\mu \widehat{\chi}^\nu V_{\mu\theta} W_{\nu\delta} \tilde{U}^{\theta\delta}) \tag{A.24}$$

$$\psi^T C \gamma_a \gamma_b \gamma_c \chi V_a A_{bc} = -2\sqrt{2}(\widehat{\psi}^\mu \widehat{\chi}^\nu V_{\mu\lambda} + \psi_\mu \chi_\lambda \widetilde{V}^{\mu\nu}) A_\nu^\lambda \tag{A.25}$$

C. SO(4)

Two vectors :

$$V_{\tilde{\alpha}} W_{\tilde{\alpha}} = -V^{\alpha\dot{\alpha}} W_{\alpha\dot{\alpha}} \tag{A.26}$$

Two antisymmetric tensors :

$$A_{\tilde{\alpha}\tilde{\beta}} B_{\tilde{\alpha}\tilde{\beta}} = 2(\vec{A}_R.\vec{B}_R + \vec{A}_L.\vec{B}_L) \quad , \quad \epsilon_{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}\tilde{\delta}} A_{\tilde{\alpha}\tilde{\beta}} B_{\tilde{\gamma}\tilde{\delta}} = 4(\vec{A}_R.\vec{B}_R - \vec{A}_L.\vec{B}_L) \tag{A.27}$$

Three antisymmetric tensors :

$$\begin{aligned} A_{\tilde{\alpha}\tilde{\beta}} B_{\tilde{\beta}\tilde{\gamma}} C_{\tilde{\gamma}\tilde{\alpha}} &= -\frac{1}{\sqrt{2}}\{A_{(R)}{}^{\dot{\alpha}\dot{\beta}} B_{(R)\ \dot{\beta}}{}^{\dot{\gamma}} C_{(R)\ \dot{\gamma}\dot{\alpha}} + A_{(L)}^{\alpha\beta} B_{(L)\ \beta}{}^{\gamma} C_{(L)\ \gamma\alpha}\} \\ &= \sqrt{2}\{\vec{A}_R \cdot (\vec{B}_R \times \vec{C}_R) + \vec{A}_L \cdot (\vec{B}_L \times \vec{C}_L)\} \end{aligned} \tag{A.28}$$

Two vectors and an antisymmetric tensor :

$$V_{\tilde{\alpha}} W_{\tilde{\beta}} A_{\tilde{\alpha}\tilde{\beta}} = \frac{1}{\sqrt{2}}\{V^{\alpha\dot{\alpha}} W_\alpha{}^{\dot{\beta}} A_{(R)\ \dot{\alpha}\dot{\beta}} + V^{\alpha\dot{\alpha}} W^\beta{}_{\dot{\alpha}} A^{(L)}_{\alpha\beta}\} \tag{A.29}$$

When the indices are contracted with the invariant tensor of SO(4) :

$$\epsilon_{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}\tilde{\delta}} V_{\tilde{\alpha}} W_{\tilde{\beta}} A_{\tilde{\gamma}\tilde{\delta}} = \sqrt{2}\{V^{\alpha\dot{\alpha}} W_\alpha{}^{\dot{\beta}} A_{(R)\ \dot{\alpha}\dot{\beta}} - V^{\alpha\dot{\alpha}} W^\beta{}_{\dot{\alpha}} A^{(L)}_{\alpha\beta}\} \tag{A.30}$$

Two traceless symmetric tensors :

$$\widehat{S}_{\tilde{\alpha}\tilde{\beta}} \widehat{R}_{\tilde{\alpha}\tilde{\beta}} = \widehat{S}^{\alpha\beta,\dot{\alpha}\dot{\beta}} \widehat{R}_{\alpha\beta,\dot{\alpha}\dot{\beta}} \tag{A.31}$$

Three symmetric tensors :

$$\widehat{S}_{\tilde{\alpha}\tilde{\beta}} \widehat{R}_{\tilde{\beta}\tilde{\gamma}} \widehat{T}_{\tilde{\gamma}\tilde{\alpha}} = -\widehat{S}^{\alpha\beta,\dot{\alpha}\dot{\beta}} \widehat{R}_{\beta\gamma,\dot{\beta}\dot{\gamma}} \widehat{T}^{\gamma}_{\alpha}{}^{,\dot{\gamma}}{}_{\dot{\alpha}} \tag{A.32}$$

Two vectors and a symmetric tensor :

$$V_{\tilde{\alpha}} W_{\tilde{\beta}} \widehat{S}_{\tilde{\alpha}\tilde{\beta}} = V^{\alpha\dot{\alpha}} W^{\beta\dot{\beta}} \widehat{S}_{\alpha\beta,\dot{\alpha}\dot{\beta}} \tag{A.33}$$

Two antisymmetric and one symmetric tensor :

$$A_{\tilde{\alpha}\tilde{\beta}} B_{\tilde{\beta}\tilde{\gamma}} \widehat{S}_{\tilde{\gamma}\tilde{\alpha}} = -\frac{1}{2}\{A^{\dot{\alpha}\dot{\beta}}_{(R)} B^{\alpha\beta}_{(L)} + A^{\alpha\beta}_{(L)} B^{\dot{\alpha}\dot{\beta}}_{(R)}\} \widehat{S}_{\alpha\beta,\dot{\alpha}\dot{\beta}} \tag{A.34}$$

One antisymmetric and two symmetric :

$$A_{\tilde{\alpha}\tilde{\beta}} \widehat{S}_{\tilde{\beta}\tilde{\gamma}} \widehat{R}_{\tilde{\gamma}\tilde{\alpha}} = -\frac{1}{\sqrt{2}}(A^{(R)}_{\dot{\alpha}\dot{\beta}} \widehat{S}^{\gamma\dot{\beta}\dot{\gamma}}_{\alpha} + A^{(L)}_{\alpha\beta} \widehat{S}^{\gamma\beta\dot{\gamma}}_{\dot{\alpha}}) \widehat{R}^{\alpha\dot{\alpha}}_{\gamma\dot{\gamma}} \tag{A.35}$$

D. SO(4) Invariants with Spinors

For the SO(4) sector $C \equiv C_2^{(2)}$

$$\psi^T C \gamma_{\tilde{\alpha}} \chi V_{\tilde{\alpha}} = \sqrt{2}(\psi_\alpha \chi_{\dot{\alpha}} - \psi_{\dot{\alpha}} \chi_\alpha) V^{\alpha\dot{\alpha}} \tag{A.36}$$

$$\psi^T C \gamma_{\tilde{\alpha}} \gamma_{\tilde{\beta}} \chi V_{\tilde{\alpha}} W_{\tilde{\beta}} = 2\psi_\alpha \chi_\beta V^{\alpha\dot{\alpha}} W^{\beta}_{\dot{\alpha}} - 2\psi_{\dot{\alpha}} \chi_{\dot{\beta}} V^{\alpha\dot{\alpha}} W^{\dot{\beta}}_{\alpha} \tag{A.37}$$

$$\psi^T \begin{Bmatrix} C_2^{(2)} \\ C_1^{(2)} \end{Bmatrix} \gamma_{\tilde{\alpha}} \gamma_{\tilde{\beta}} \chi A_{\tilde{\alpha}\tilde{\beta}} = -2\sqrt{2}\{A^{\dot{\alpha}\dot{\beta}} \psi_{\dot{\alpha}} \chi_{\dot{\beta}} \mp A^{\alpha\beta} \psi_\alpha \chi_\beta\} \tag{A.38}$$

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